Topic 5: Additive structure in value sets of norm forms

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This project is in number theory, with a special focus on the arithmetic of norm forms. The flavour can be algebraic, analytic, combinatorial, or a combination of all of those, depending on the student's background and interests.

The investigation of additive structure in sets of arithmetic interest, such as primes, squares, values of quadratic forms, or more generally, of multivariate polynomials, is a classical topic in number theory. For example, the value sets S_F of integral binary quadratic forms

$$F = ax^2 + bxy + cy^2$$

have been studied from many different points of view, starting with Fermat, Euler, Lagrange and Gauß. In recent years, the area has seen an enormous uptake (e.g. [BD11, BMS14, DEK$^+$22]) due to breakthroughs in related areas of analytic number theory, harmonic analysis and additive combinatorics [GT08, May16]. Amongst the investigated properties are density, the set of prime values, gaps between values of the elements in S_F, arithmetic progressions in these values, e.g. [DEK$^+$22, EF19]; special attention was devoted to the basic case $F = x^2 + y^2$ of sums of two squares.

Value sets of specific binary quadratic forms, such as sums of two squares, have been of particular interest due to their accessibility by analytic, combinatorial as well as algebraic methods. Generalisations to higher degree polynomials seem extremely hard in general, as most of the algebraic structure will be lost. A rich and structured intermediate class of polynomials is provided by norm forms of algebraic number fields, which should be seen as generalising binary quadratic forms to degree higher than two. Here, algebraic number theory provides a robust toolset, which can be combined with analytic methods to attack analogous additive questions for higher degree norm forms. This approach is showcased by recent work of Elsholtz and Frei [EF19], where the more general approach not only led to a full generalisation to norm forms, but also to improvements in the original result regarding arithmetic progressions represented by sums of two squares.

References

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