GENERALIZED EUCLIDEAN RINGS (GE-RINGS)

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This project about Generalized Euclidean Rings (GE-rings) uses methods from ring theory, algebraic number theory, linear algebra, and algebraic K-theory, to study rings satisfying weaker versions of Euclidean algorithm, and matrices over such rings. Following Cohn [4], a GE-ring is a ring in which every invertible matrix is a product of elementary matrices and dilations. Similarly, a ring is an ID-ring if every singular matrix is a product of idempotent matrices. GE_2-rings and ID_2-rings satisfy the respective properties for 2×2 matrices.

The property GE_2 can also be expressed as: any unimodular pair (a, b) can be transformed into $(1, 0)$ by a finite sequence of elementary operations (adding a scalar multiple of a to b or vice versa). This suggests the stronger property (called a "weak (Euclidean) algorithm"), that any pair (a, b) can be transformed into $(c, 0)$ by a series of elementary operations.

Recent results show relationships between the properties GE and ID [9, 2], in particular, for Bézout-rings (rings in which every finitely generated ideal is principal), the properties GE_2, GE, ID_2, and ID, and the existence of a weak algorithm, are all equivalent [8]. Many questions remain open, however, especially in the case of rings that are not a priori Bézout. For instance, for integral domains, does ID_2 imply Bézout? Similarly, there are classical sufficient conditions and necessary conditions for both GE and ID [5, 7, 6, 3], but a general characterization remains elusive. It is known which rings of integers in quadratic number fields satisfy GE_2, but it is an open problem which ones satisfy ID_2 [1].

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