ON α-GREEDY EXPANSIONS OF NUMBERS

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Abstract. We study a redundant binary number system that was recently introduced by Székely and Wang. It works recursively on a natural number n as follows: let k satisfy $2^k \leq \frac{2}{3}n < 2^{k+1}$; then 2^k is subtracted from n, and the expansion continues. It stops, when a power of 2 is reached.

For this and more general number systems, where the factor $2/3$ is replaced by a general one, we find an explicit formula for the kth digit $\varepsilon_k \in \{0, 1, 2\}$. This allows us to compute the cumulative frequency of a given digit, among the first N integers. A method of Delange produces not only the leading term of order $N \log N$, but also the fluctuating term of order N, and the Fourier coefficients of the periodic functions that are involved.

Furthermore, we can compute the redundant expansions from *right-to-left*, by translating the ordinary binary expansion using a (finite state) transducer, provided the factor (such as $2/3$) is *rational.*

1. Introduction

Every (positive) integer has a unique representation in base 2 with digits 0 or 1. If one, however, allows *more* digits, like $\{-1, 0, 1\}$, then one is in the area of *redundant number systems*; representations are (in general) no longer unique, and one has some freedom to choose the most convenient ones.

Reitwiesner [13] came up with the *non-adjacent form*, which never has adjacent nonzero digits. This is useful in computer arithmetic. We refer to Knuth [12] for more details.

More recently, such redundant expansions became relevant in *Cryptography*, because a small so-called *Hamming weight* results in fast computations of (high) scalar multiples nP in Abelian groups such as the point group of an elliptic curve.

Other computer science applications include *jump trees* [9], and *mergesort* [4], just to name of few.

A recent survey about *numeration systems* is [7]; compare also [15].

Recently, Székely and Wang [16, 17] invented a *novel* redundant binary number system when studying trees with a *large* number of subtrees: Let k be defined by $2^k \leq \frac{2}{3}n < 2^{k+1}$; then 2^k is subtracted from n, and the expansion continues. It stops, when a power of 2 is reached.

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This leads to digits in the set $\{0,1,2\}$. Of course, one can generalise this definition readily by replacing the factor $2/3$ by $1/\alpha$; (it is for convenience that we use $1/\alpha$ instead of just α). Clearly, $\alpha = 1$ just produces the traditional binary number system.

We study these expansions in this paper and call them *α-greedy expansions.* As it will become clear in the sequel, the reasonable range for the parameter α is $1 \le \alpha \le 3/2$. This leads to expansions with digits $\{0,1,2\}$, and larger digits are computed before smaller digits, i.e., the recursive computation of the digits works from *left to right*.

We will first find a method to compute the digits in a non-recursive fashion, by establishing a *formula* for the digits. In short, one has to look at $n/2^{k+1} \pmod 1$. The unit interval is split into three (unions of) intervals, except for some exceptional points. According to which interval is hit, the outcome is one of the digits $0,1,2$.

This leads to the natural question about the frequency of the digits. Basically, this depends on the respective lengths of the above-mentioned intervals. Digit 1 always gets $1/2$, whereas 0 and 2 get $1-\alpha/2$ and $(\alpha-1)/2$, respectively. In order to get precise results of this rough estimate, we use an idea of Delange [6]. We count the number of occurrences of digit 1 resp. 2 in all the integers $0,1,\dots,N-1$; from this information, one can also count the digit 0 (one must make, however, some conventions, because of possible *leading zeros*). The result is the (expected) leading term $\lambda_d N \log_2 N$, but the next term $N\Phi_d(\log_2 N)$ with a periodic function $\Phi_d(x)$ (continuous, period 1) is perhaps less expected. We are able to compute the Fourier coefficients of these periodic functions; they involve the Hurwitz ζ-function, evaluated at some special values.

The *explicit* digit formulæ lead to a convenient method to compute the expansion from the ordinary binary expansion by *translation.* This translation is performed by (finite state-) transducers, which work from right to left. They are described in general for *rational* α, (as they do not exist for irrational α,) and explicitly drawn for several important special cases. By inspection, one sees then, that the expansion of Székely and Wang can be obtained as follows: If the binary expansion of n is canonically written as $10^{a_1}10^{a_2}\dots10^{a_s}$, then the last group 10^{a_s} is left as it stands, but every other one is replaced by $10^a \longrightarrow 01^{a-1}2$ if $a \ge 1$ and by 1 otherwise.

Remark 1.1. The *number $f(n)$ of representations* of an integer n as $\sum_j \varepsilon_j 2^j$, with $\varepsilon_j \in \{0,1,2\}$, was determined by Reznick [14], compare also [2]: it is given by the recursive formula

$$f(2n+1) = f(n), \qquad f(2n+2) = f(n) + f(n+1), \qquad f(0) = 1.$$

2. Digit Formula

We use Iverson's notation

$$[\mathit{condition}] = \begin{cases} 1, & \text{if } \mathit{condition} \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $1 \le \alpha \le 3/2$ be fixed throughout the paper.

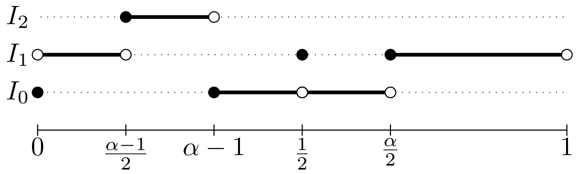

FIGURE 1. Characteristic sets I_0, I_1, and I_2 for $\alpha = 4/3$.

We define the α-greedy-expansion $\boldsymbol{\varepsilon}(n) = (\varepsilon_j(n))_{j\geq 0}$ of a positive integer n as follows: If n equals 2^k for some integer $k \geq 0$, we set $\varepsilon_j(n) = [j = k]$ for $j \geq 0$. Otherwise, we choose the unique integer k satisfying

$$2^k \leq \frac{1}{\alpha} n < 2^{k+1} \tag{1}$$

and set

$$\varepsilon_j(n) = [j = k] + \varepsilon_j(n - 2^k) \tag{2}$$

for $j \geq 0$. Since $0 < n - 2^k < n$, this defines $\varepsilon_j(n)$ uniquely.

It is an immediate consequence of (2) that $\boldsymbol{\varepsilon}(n)$ is indeed a binary expansion of n, i.e.,

$$\mathsf{value}(\boldsymbol{\varepsilon}(n)) := \sum_{j\geq 0} \varepsilon_j(n) 2^j = n.$$

Note that the special case $\alpha = 1$ exactly yields the standard binary expansion of n. The case $\alpha = 3/2$ has been considered by Székely and Wang [17].

Theorem 1. *Let n be a positive integer and j be a nonnegative integer. We set*

$$\begin{aligned} I_0 &:= \{0\} \cup [\alpha - 1, 1/2) \cup (1/2, \alpha/2), \\ I_1 &:= (0, (\alpha - 1)/2) \cup \{1/2\} \cup [\alpha/2, 1), \\ I_2 &:= [(\alpha - 1)/2, \alpha - 1). \end{aligned}$$

Then the following holds:

1. *If $n/2^{j+1} < \alpha - 1$, then $\varepsilon_j(n) = 0$.*
2. *If $n/2^{j+1} \geq \alpha - 1$ and $\{n/2^{j+1}\} \in I_\eta$ for some $\eta \in \{0, 1, 2\}$, then $\varepsilon_j(n) = \eta$.*

Here, $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$ of a real number x.

We note that for our choice of α, we have

$$0 \leq \frac{\alpha - 1}{2} \leq \alpha - 1 \leq 1/2 \leq \alpha/2 \leq 3/4,$$

and $I_0 \cup I_1 \cup I_2 = [0, 1)$, thus the theorem allows to compute all digits of $\boldsymbol{\varepsilon}(n)$. In particular, the digits used are $\{0, 1, 2\}$ except when $\alpha = 1$, where, of course, only the digits $\{0, 1\}$ are used. The sets I_η for $\alpha = 4/3$ are shown in Figure 1.

The following simple lemma shows that the assumption $\alpha \leq 3/2$ makes the α-expansion behave somewhat more regularly.

Lemma 2.1. *Let n not be a power of 2 and $2^j \leq \frac{n}{\alpha} < 2^{j+1}$. If $n - 2^j$ is a power of 2, then $n - 2^j \leq 2^{j-1}$.*

This means that the contributions to $\varepsilon_j(n)$ for arbitrary n and j either entirely come from (1) or entirely from a power of 2, but it cannot occur that contributions to the same digit come from both cases.

Proof of Lemma 2.1. We have

$$n - 2^j < (2\alpha - 1)2^j \leq 2^{j+1}.$$

Since $n - 2^j$ has been assumed to be a power of 2 and $n - 2^j = 2^j$ would be a contradiction to the assumption that n is not a power of 2, the assertion of the lemma is proved. □

Remark 2.2. For $\alpha > 3/2$, Lemma 2.1 does not hold. As an example, consider the case $\alpha = 3$ and consider the expansion of 13:

$$\begin{aligned}
&2^2 \leq 13/3 < 2^3, && 13 = 2^2 + 9,\\
&2^1 \leq 9/3 < 2^2, && 13 = 2^2 + 2^1 + 7,\\
&2^1 \leq 7/3 < 2^2, && 13 = 2^2 + 2^1 + 2^1 + 5,\\
&2^0 \leq 5/3 < 2^1, && 13 = 2^2 + 2^1 + 2^1 + 2^0 + 4,\\
&2^2 = 4, && 13 = 2^2 + 2^1 + 2^1 + 2^0 + 2^2.
\end{aligned}$$

Note that an additional summand 2^2 occurs at the end, although the process has already reached summands 2^0.

One could still formulate digit formulæ, however, they would require more "look-ahead" expressed in more exceptional points (such as 0 and 1/2 in Theorem 1). It seems inadequate to deal with these technical difficulties within the frame of this paper.

One might also want to change the special treatment of powers of 2. The rule considered here has the advantage that divisibility by powers of 2 is reflected by the corresponding number of trailing zeros. Just note that some kind of terminating rule is necessary in order to stop the process anyway.

Proof of Theorem 1. If $n = 2^K$ for some integer K, the assertions of the theorem immediately follow from the definitions. Therefore, we can exclude this case in the following.

We choose the integer J such that $\alpha - 1 \leq n/2^{J+1} < 2(\alpha - 1) \leq \alpha$. If $j > J$, we have $n/2^{j+1} < \alpha - 1$ and therefore $n < 2^j$, which means that a summand 2^j cannot possibly occur and we have $\varepsilon_j(n) = 0$.

For real x, we define $r(x)$ to be the unique number in the interval $[\alpha - 1, \alpha)$ such that $r(x) - x$ is an integer. We set

$$\begin{aligned}
J_0 &:= [\alpha - 1, 1/2) \cup (1/2, \alpha/2) \cup \{1\},\\
J_1 &:= \{1/2\} \cup [\alpha/2, 1) \cup (1, (\alpha + 1)/2),\\
J_2 &:= [(\alpha + 1)/2, \alpha).
\end{aligned}$$

Then it is clear that $r(x) \in J_\eta$ if and only if $\{x\} \in I_\eta$ for $\eta \in \{0, 1, 2\}$.

Let K be maximal such that 2^K divides n and set

$$n_j := n - \sum_{k=j+1}^{J} \varepsilon_k(n) 2^k \tag{3}$$

for $J \geq j \geq 0$. We now prove the assertions of the theorem by backwards induction for $J \geq j \geq K$. As an additional induction hypothesis, we assume that

$$\alpha - 1 \leq \frac{n_j}{2^{j+1}} < \alpha, \qquad \frac{n_j}{2^{j+1}} \neq 1, \tag{4}$$

which, by definition, holds for $j = J$. From the definition of n_j and (4) we immediately see that $n_j/2^{j+1} = r(n/2^{j+1})$.

We first consider the case that n_j is not a power of 2, thus $n_j/2^{j+1} \notin \{1/2, 1\}$.

If $n_j/2^{j+1} \in J_0$, we get $n_j/\alpha < 2^j$, i.e., there is no contribution to $\varepsilon_j(n)$ coming from (1). On the other hand, the next digit has to come from (1) since n_j is not a power of 2, whence $\varepsilon_j(n) = 0$ by Lemma 2.1. Furthermore, we get $\alpha - 1 \leq n_{j-1}/2^j = n_j/2^j < \alpha$, i.e., hypothesis (4) for $j - 1$.

If $n_j/2^{j+1} \in J_1$, we conclude that

$$2^j \leq \frac{1}{\alpha} n_j < 2^{j+1} \quad \text{and} \quad \frac{\alpha - 1}{\alpha} 2^j \leq \frac{n_j - 2^j}{\alpha} < 2^j,$$

thus $\varepsilon_j(n) = 1$ and $n_{j-1} = n_j - 2^j$, where Lemma 2.1 has been used. From this we easily get (4) for $j - 1$.

If $n_j/2^{j+1} \in J_2$, we obtain

$$2^j \leq \frac{1}{\alpha}(n_j - 2^j) < \frac{1}{\alpha} n_j < 2^{j+1} \quad \text{and} \quad \frac{\alpha - 1}{\alpha} 2^j \leq \frac{n_j - 2 \cdot 2^j}{\alpha} < \frac{\alpha - 1}{\alpha} 2^{j+1} \leq 2^j.$$

Using Lemma 2.1, we conclude that $\varepsilon_j(n) = 2$ and $n_{j-1} = n_j - 2 \cdot 2^j$. Again, the induction hypothesis (4) also holds for $j - 1$, since $n_j - 2 \cdot 2^j$ cannot equal 2^j.

Finally, we consider the case $n_j = 2^\ell$ for some ℓ and we get $\ell \leq j$ from (4) and $\ell = K$ from (3). We see that $\varepsilon_j(n) = \cdots = \varepsilon_{K+1}(n) = 0$ as well as $n_j = \cdots = n_K$ and $n_j/2^{j+1}$, $\ldots$, $n_{K+1}/2^{K+2} \in J_0$. Next we obtain $\varepsilon_K(n) = 1$, $n_K/2^{K+1} = 1/2 \in J_1$, and $n_{j'} = 0$ and therefore $\varepsilon_{j'}(n) = 0$ for $j' < K$. Since $n/2^{j'+1}$ is an integer for $j' < K$, we also get $r(n/2^{j'+1}) = 1 \in J_0$ for those j'. □

Remark 2.3. Values $\alpha < 1$ lead to negative values $n - 2^k$. The definition has to be modified in such a way that for negative n, negative digits are allowed. In order to obtain an analogue of Lemma 2.1, one has to require that $\alpha \geq 1/2$. The case of $\alpha = 2/3$ is known as the Non-Adjacent-Form (cf. Reitwiesner [13] and Heuberger [10]). Digit formulæ can be derived for $\alpha \in [1/2, 2/3]$. For α not in this range, this needs not to be the case. As an example, consider $\alpha = 3/4$, $x_m = (2^{2m+2} - 1)/3$ and $y_m = x_m + 2^{2m+1}$ whose 3/4-expansion differs in the third digit from the right. Thus in this case, there cannot be a digit formula only involving fractional parts of $n/2^{k+\ell}$ for some constant ℓ.

3. Counting Digits

The aim of this section is to compute the frequency of the digits in α-greedy-expansions. To this aim, we use Delange's [6] method and the digit formulæ given in Theorem 1. The case of the standard binary expansion ($\alpha = 1$) has been dealt with in Delange [6] and is excluded here for technical reasons.

Theorem 2. *Let $1 < \alpha \leq 3/2$, N be a positive integer and $d \in \{1, 2\}$. Then there is a continuous 1-periodic function Φ_d such that the number $S_d(N)$ of occurrences of the digit d in the α-greedy-expansions of the positive integers less than N can be calculated as*

$$S_d(N) := \sum_{n=1}^{N-1} \sum_{k\geq 0} [\varepsilon_k(n) = d] = \lambda_d N \log_2 N + N\Phi_d(\log_2 N) + O(\log N),$$

where

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_2 = \frac{\alpha - 1}{2}.$$

The periodic function Φ_d has a uniformly convergent Fourier series, the Fourier coefficients $c_n^{(d)} = \int_0^1 \Phi_d(x) \exp(-2\pi i n x)\, dx$, $n \in \mathbb{Z}$, are given by

$$\begin{aligned}
c_0^{(1)} &= \frac{3}{4} - \frac{1}{2\log 2} + \log_2 \Gamma\left(\frac{\alpha}{2}\right) - \log_2 \Gamma\left(\frac{\alpha-1}{2}\right) - \log_2(\alpha - 1),\\
c_n^{(1)} &= \frac{\zeta\left(\chi_n, \frac{\alpha}{2}\right) - \zeta\left(\chi_n, \frac{\alpha-1}{2}\right) + (\alpha-1)^{-\chi_n}}{(1+\chi_n)\chi_n \log 2}, \qquad n \neq 0,\\
c_0^{(2)} &= -\frac{\alpha+3}{4} - \frac{\alpha-1}{2\log 2} + \frac{1}{2}\log_2 \pi - \log_2 \Gamma\left(\frac{\alpha}{2}\right) + \sum_{j\geq 1} j\eta_j 2^{-j-1},\\
c_n^{(2)} &= \frac{\zeta\left(\chi_n, \frac{\alpha-1}{2}\right) - \zeta(\chi_n, \alpha - 1)}{(1+\chi_n)\chi_n \log 2}, \qquad n \neq 0,
\end{aligned} \tag{5}$$

where $2 - \alpha = \sum_{j\geq 1} \eta_j 2^{-j}$ is the standard binary expansion of $2 - \alpha$ (in case of ambiguity, choose the expansion with finitely many digits 1), $\zeta(s, a)$ denotes the Hurwitz Zeta function, defined for $\operatorname{Re} s > 1$ by $\zeta(s, a) := \sum_{k\geq 0}(k + a)^{-s}$, and $\chi_n = 2\pi i n/\log 2$ for $n \in \mathbb{Z}$.

As usual, the digit 0 is not dealt with explicitly in order to avoid dealing with leading zeros.

In Figure 2, the periodic function and the values approximated by it are displayed. As predicted, for growing N, the fit becomes better and better. The periodic function has been plotted using about 4000 Fourier coefficients.

The following lemma summarises those parts of the computation which are quite independent of our digit system.

Lemma 3.1. *Let $H \in [0, 1)$ be a measurable set, $s < 1$ be a constant such that*

$$M_k := \#\left\{a \in \mathbb{Z} : \left[\frac{a}{2^k}, \frac{a+1}{2^k}\right) \cap \partial H \neq \emptyset\right\} = O(2^{sk}),$$

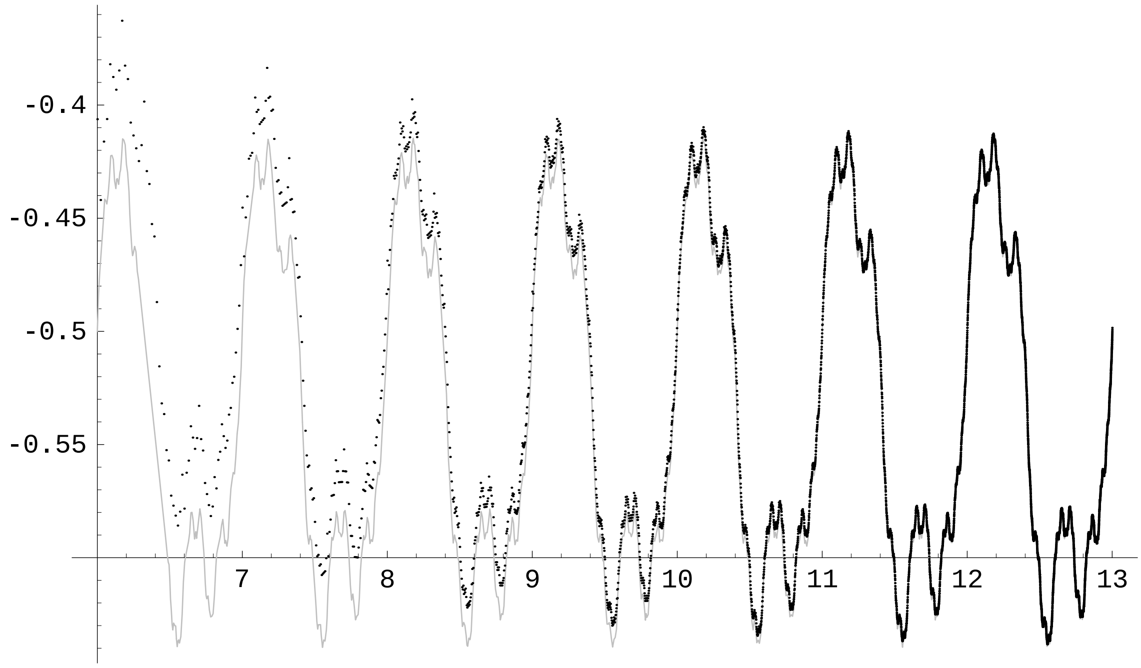

FIGURE 2. The periodic function $\Phi_1(\log_2 N)$ (continuous gray line) and $1/N(S_1(N) - \frac{1}{2}N\log_2 N)$ (black dots) for $2^6 \le N \le 2^{13}$ and $\alpha = 3/2$. The N-axis is scaled logarithmically.

where ∂H is the boundary of H, and c be a nonnegative integer. Then

$$\sum_{n=0}^{N-1} \sum_{k=0}^{\lfloor \log_2 N \rfloor + c} \left[\left\{ \frac{n}{2^{k+1}} \right\} \in H \right] = \lambda(H) N \log_2 N + N\Phi_{H,c}(\log_2 N) + O(N^s + \log N),$$

where $\lambda(H)$ is the Lebesgue measure of H and $\Phi_{H,c}$ is a 1-periodic function, continuous in the open interval $(0,1)$. Its Fourier coefficients $c_n = \int_0^1 \Phi_{H,c}(x)\exp(-2\pi i n x)\,dx$, $n \in \mathbb{Z}$, are given by

$$\begin{aligned} c_n =& \Big(-\frac{1}{2}\lambda(H) + \sum_{k\ge 0} \beta_k \Big) [n=0] + \frac{\lambda(H)}{\chi_n \log 2} [n \ne 0] \\ &+ \frac{1}{(1+\chi_n)\log 2} \Big(-\lambda(H) + 2^{c+1}\lambda(H \cap [0, 2^{-c-1}]) + \int_{H\cap[2^{-c-1},1]} y^{-(1+\chi_n)}\,dy \Big) \qquad (6) \\ &+ \frac{1}{(1+\chi_n)\log 2} \sum_{k\ge 1} \int_0^1 (y+k)^{-(1+\chi_n)} \left([y \in H] - \lambda(H) \right) dy, \end{aligned}$$

where

$$\beta_k = \int_0^1 \left(\left[\frac{\lfloor 2^{k+1} y \rfloor}{2^{k+1}} \in H \right] - [y \in H] \right) dy, \qquad k \ge 0.$$

Proof of Lemma 3.1. The first part follows along the lines of the proofs of Theorem 17 in [11] and Theorem 5 in [8]. We get

$$\Phi_{H,c}(x) = \lambda(H)(1 + c - x) + \Psi_{H,c}(x) + \sum_{k\geq 0} \beta_k,$$

where

$$\Psi_{H,c}(x) = \sum_{k\geq 0} 2^{-(x+k-c-1)} \int_0^{2^{x+k-c-1}} ([\{y\} \in H] - \lambda(H))\, dy$$

for $x \in [0, 1)$ and consider $\Phi_{H,c}$ and $\Psi_{H,c}$ as 1-periodic functions. The error term is bounded by $O(\log N)$ if $s = 0$.

We want to compute the Fourier coefficients c_n, $n \in \mathbb{Z}$, of $\Phi_{H,c}(x)$. Denoting the Fourier coefficients of $\Psi_{H,c}$ by d_n, $n \in \mathbb{Z}$, we easily get

$$c_n = \left(\left(c + \frac{1}{2}\right)\lambda(H) + \sum_{k\geq 0} \beta_k\right) [n = 0] + \frac{\lambda(H)}{\chi_n \log 2} [n \neq 0] + d_n. \tag{7}$$

We first rewrite $\Psi_{H,c}(x)$ as

$$\Psi_{H,c}(x) = \sum_{\ell \geq -c-1} 2^{-(x+\ell)} \int_0^{2^{x+\ell}} ([\{y\} \in H] - \lambda(H))\, dy$$

and note that the lower bound of the integral can be replaced by any integer less than $2^{x+\ell}$ without changing its value. Thus the integral is bounded by 2 and the sum converges uniformly for $x \in [0, 1]$.

By definition and uniform convergence, we have

$$d_n = \int_0^1 \Psi_{H,c}(x) 2^{-\chi_n x}\, dx = \sum_{\ell \geq -c-1} \int_0^1 2^{-(x+\ell)-\chi_n x} \int_0^{2^{x+\ell}} ([\{y\} \in H] - \lambda(H))\, dy dx.$$

Replacing $x + \ell$ by x, collecting the contributions of $x < 0$ and splitting the contributions for $x > 0$ into suitable parts for the fractional part yields

$$\begin{aligned} d_n &= \sum_{\ell \geq -c-1} \int_\ell^{\ell+1} 2^{-(1+\chi_n)x} \int_0^{2^x} ([\{y\} \in H] - \lambda(H))\, dy dx \\ &= \int_{-c-1}^0 2^{-(1+\chi_n)x} \int_0^{2^x} ([y \in H] - \lambda(H))\, dy dx \\ &\quad + \sum_{k\geq 1} \int_{\log_2 k}^{\log_2(k+1)} 2^{-(1+\chi_n)x} \int_k^{2^x} ([\{y\} \in H] - \lambda(H))\, dy dx. \end{aligned}$$

We calculate the easy part of the first integral and swap the order of integration in the remaining integrals to obtain

$$\begin{aligned} d_n = &-\lambda(H)\int_{-c-1}^{0} 2^{-(1+\chi_n)x}\int_0^{2^x} dydx \\ &+\int_0^{2^{-c-1}}\int_{-c-1}^{0}[y\in H]\,2^{-(1+\chi_n)x}\,dxdy+\int_{2^{-c-1}}^{1}\int_{\log_2 y}^{0}[y\in H]\,2^{-(1+\chi_n)x}\,dxdy \\ &+\sum_{k\geq 1}\int_k^{k+1}\int_{\log_2 y}^{\log_2(k+1)} 2^{-(1+\chi_n)x}\left([\{y\}\in H]-\lambda(H)\right)dxdy. \end{aligned}$$

We perform all possible integrations, note that $\int_k^{k+1}([y\in H]-\lambda(H))\,dy$ vanishes, and cancel out some terms and obtain

$$\begin{aligned} d_n = &-[n=0]\,\lambda(H)(c+1) \\ &+\frac{1}{(1+\chi_n)\log 2}\left(-\lambda(H)+2^{c+1}\lambda(H\cap[0,2^{-c-1}])+\int_{H\cap[2^{-c-1},1]} y^{-(1+\chi_n)}\,dy\right) \\ &+\frac{1}{(1+\chi_n)\log 2}\sum_{k\geq 1}\int_0^1 (y+k)^{-(1+\chi_n)}\left([y\in H]-\lambda(H)\right)dy. \end{aligned}$$

Together with (7), we get (6). □

Proof of Theorem 2. We first consider the case $d=2$. For positive n, there is exactly one k such that

$$\frac{n}{2^{k+1}}\in I_2$$

(no fractional part!) and by Theorem 1, we do not have $\varepsilon_k(n)=2$ for this k, since $n/2^{k+1}<\alpha-1$. We choose $c:=\lfloor -\log_2(\alpha-1)\rfloor+1$, set $K:=\lfloor\log_2 N\rfloor+c$ implying that $n/2^{k+1}<(\alpha-1)/2$ for all $k>K$. Thus we get

$$\begin{aligned} \sum_{n=1}^{N-1}\sum_{k\geq 0}[\varepsilon_k(n)=2] &= \sum_{n=1}^{N-1}\sum_{k=0}^{K}[\varepsilon_k(n)=2] = \sum_{n=0}^{N-1}\sum_{k=0}^{K}\left[\left\{\frac{n}{2^{k+1}}\right\}\in I_2\right]-(N-1) \\ &= \lambda(I_2)N\log_2 N+N\big(\Phi_{I_2,c}(\log_2 N)-1\big)+O(\log_2 N) \end{aligned}$$

by Lemma 3.1 for $H=I_2$, since $M_k\leq 2$. We note that $\lambda(I_2)=(\alpha-1)/2$ and set $\Phi_2(x)=\Phi_{I_2,K-\lfloor\log_2 N\rfloor}(x)-1$. We note that by definition, we have $S_d(2^L)-S_d(2^L-1)=O(L)$ and therefore $\Phi_2(0)-\Phi_2(1-\log_2(1-2^{-L}))=O(L2^{-L})$. Thus $\Phi_2(1)=\Phi_2(0)$ by continuity. Hence Φ_2 is a 1-periodic continuous function.

We now compute the Fourier coefficients using (6). Note that $2^{-c-1} < (\alpha-1)/2$. Therefore

$$
\begin{aligned}
c_n^{(2)} &= [n=0]\Bigl(-\frac{\alpha-1}{4}-1+\sum_{k\ge 0}\beta_k\Bigr)+[n\neq 0]\,\frac{\alpha-1}{2\chi_n\log 2}\\
&\quad+\frac{1}{(1+\chi_n)\log 2}\Bigl(-\frac{\alpha-1}{2}+g_n(\alpha-1)-g_n\Bigl(\frac{\alpha-1}{2}\Bigr)\Bigr)\\
&\quad+\frac{1}{(1+\chi_n)\log 2}\sum_{k\ge 1}\Bigl(g_n(k+\alpha-1)-g_n\Bigl(k+\frac{\alpha-1}{2}\Bigr)-\frac{\alpha-1}{2}(g_n(k+1)-g_n(k))\Bigr),
\end{aligned}
$$

where

$$
g_n(y)=\begin{cases}\log y, & \text{if } n=0,\\ -\frac{y^{-\chi_n}}{\chi_n}, & \text{if } n\neq 0.\end{cases}
$$

We obtain

$$
c_0^{(2)} = -\frac{\alpha-1}{2}\Bigl(\frac12+\frac{1}{\log 2}\Bigr)-1+\log_2\Gamma\Bigl(\frac{\alpha-1}{2}\Bigr)-\log_2\Gamma(\alpha-1)+\sum_{k\ge 0}\beta_k
$$

and

$$
c_n^{(2)} = \frac{1}{(1+\chi_n)\chi_n\log 2}\Bigl(\zeta\Bigl(\chi_n,\frac{\alpha-1}{2}\Bigr)-\zeta(\chi_n,\alpha-1)\Bigr)
$$

for $n\neq 0$. Note that $\zeta(\chi_n,a)=O(\sqrt{n}\,)$ (cf. Whittaker and Watson [18, § 13.51]), thus the Fourier series is uniformly convergent. Since Φ_2 is continuous, the Fourier series converges pointwise to Φ_2 by Fejér's theorem.

We still have to compute $\sum\beta_k$. For $k\ge 0$, we have

$$
\begin{aligned}
\beta_k &= -\frac{\alpha-1}{2}+\sum_{0\le a<2^{k+1}}\frac{\bigl[(\alpha-1)2^k\le a<(\alpha-1)2^{k+1}\bigr]}{2^{k+1}}\\
&=\frac{\lceil(\alpha-1)2^{k+1}\rceil}{2^{k+1}}-\frac{\lceil(\alpha-1)2^{k}\rceil}{2^{k+1}}-\frac{\alpha-1}{2}=\frac{\lfloor(1-\alpha)2^{k}\rfloor}{2^{k+1}}-\frac{\lfloor(1-\alpha)2^{k+1}\rfloor}{2^{k+1}}-\frac{\alpha-1}{2}\\
&=-\eta_{k+1}2^{-k-2}+\sum_{j\ge k+2}\eta_j2^{-j-1}.
\end{aligned}
$$

Thus

$$
\sum_{k\ge 0}\beta_k=-\frac{2-\alpha}{2}+\sum_{j=2}^{\infty}\sum_{k=0}^{j-2}\eta_j2^{-j-1}=-\frac{2-\alpha}{2}+\sum_{j\ge 1}(j-1)\eta_j2^{-j-1}.
$$

Using the identity $\Gamma(2s)/(\Gamma(s)\Gamma(s+1/2))=2^{2s-1}/\sqrt{\pi}$ we get (5) in this case.

Next, we consider the case $d=1$. We set $c:=\lfloor-\log_2(\alpha-1)\rfloor$ and $K:=\lfloor\log_2 N\rfloor+c$ which implies that for $k>K$ and $n<N$, $n/2^{k+1}<(\alpha-1)$ and therefore $\varepsilon_k(n)=0$.

We get

$$\sum_{n=1}^{N-1}\sum_{k\geq 0}[\varepsilon_k(n)=1]=\sum_{n=0}^{N-1}\sum_{k=0}^{K}\left[\left\{\frac{n}{2^{k+1}}\right\}\in I_1\right]-\sum_{n=1}^{N-1}\sum_{k=0}^{K}\left[\frac{n}{2^{k+1}}<\frac{\alpha-1}{2}\right].$$

Since $(\alpha-1)2^k\leq N$ for $k\leq K$, we have

$$\begin{aligned}\sum_{n=1}^{N-1}\sum_{k=0}^{K}\left[\frac{n}{2^{k+1}}<\frac{\alpha-1}{2}\right]&=\sum_{k=0}^{K}\left\lfloor(\alpha-1)2^k\right\rfloor+O(1)=2^{K+1}(\alpha-1)+O(K)\\&=N2^{1-\{\log_2 N\}+\lfloor-\log_2(\alpha-1)\rfloor}(\alpha-1)+O(\log N).\end{aligned}$$

We apply Lemma 3.1 and use the same continuity argument as above.

For the Fourier coefficients, we note that $(\alpha-1)/2\leq 2^{-c-1}<\alpha-1$. Taking the additional term $-2^{1+c-x}(\alpha-1)$ into account, we obtain from Lemma 3.1 that

$$\begin{aligned}c_n^{(1)}=&\left(-\frac{1}{4}+\sum_{k\geq 0}\beta_k\right)[n=0]+\frac{[n\neq 0]}{2\chi_n\log 2}-\frac{(\alpha-1)2^c}{(1+\chi_n)\log 2}\\&+\frac{1}{(1+\chi_n)\log 2}\left(-\frac{1}{2}+2^c(\alpha-1)+g_n(1)-g_n\left(\frac{\alpha}{2}\right)\right)\\&+\frac{1}{(1+\chi_n)\log 2}\sum_{k\geq 1}\left(g_n\Big(k+\frac{\alpha-1}{2}\Big)-g_n\left(k+\frac{\alpha}{2}\right)+\frac{g_n(k+1)-g_n(k)}{2}\right).\end{aligned}$$

Thus we have

$$c_0^{(1)}=\frac{3}{4}-\frac{1}{2\log 2}+\log_2\Gamma\left(\frac{\alpha}{2}\right)-\log_2\Gamma\Big(\frac{\alpha-1}{2}\Big)-\log_2(\alpha-1)+\sum_{k\geq 0}\beta_k$$

and

$$c_n^{(1)}=\frac{\zeta\left(\chi_n,\frac{\alpha}{2}\right)-\zeta\left(\chi_n,\frac{\alpha-1}{2}\right)+(\alpha-1)^{-\chi_n}}{(1+\chi_n)\chi_n\log 2}$$

for $n\neq 0$. The Fourier series converges pointwise by the same observation as above.

Finally, we compute $\sum\beta_k$ in this case, too. We obtain

$$\beta_k=\frac{\left\lceil(\alpha-1)2^k\right\rceil+2^k-\left\lceil\alpha 2^k\right\rceil}{2^{k+1}}=0.$$

$\square$

Remark 3.2. The function that maps a number x written in binary as $(0.\varepsilon_1\varepsilon_2\ldots)_2$ to $\sum_{j\geq 1}j\varepsilon_j/2^j$, which appears in the computation of the Fourier coefficient $c_0^{(2)}$, is not uncommon in the literature and appears at least in [1, 3, 4, 5].

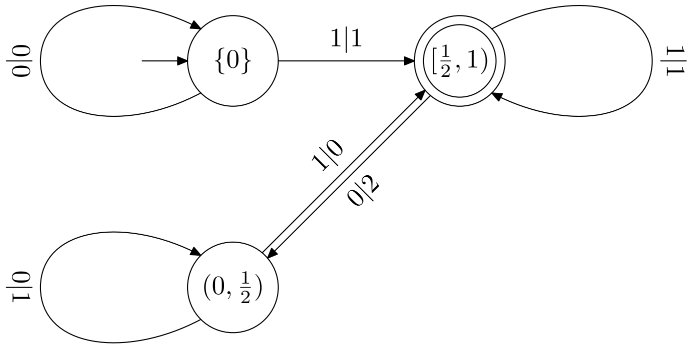

FIGURE 3. Right-To-Left-Transducer for $\alpha = 3/2$.

4. Right-To-Left Transducer

The α-greedy expansion has been defined from left to right, i.e., from the most significant digit to the least significant digit. Of course, the digit formulæ in Theorem 1 also allows us to compute the digits from right to left. The aim of this section is to investigate whether the digits can be computed from right to left from the standard binary expansion by using a transducer automaton.

As can be seen from the additional condition $n/2^{j+1} \geq \alpha - 1$ in Theorem 1, leading zeros are not quite natural in the α-greedy expansions. Therefore, we do not allow leading zeros in the standard binary expansions of the input to our transducers.

We prove the following theorem.

Theorem 3. *The following two assertions are equivalent.*

(1) *There is a finite deterministic transducer automaton rewriting the standard binary expansion* $(1, b_{L-1}, \ldots, b_0)$ *of positive integers to the α-greedy expansion of the same integer from right to left.*
(2) *The number α is rational.*

In this case there exists such a transducer automaton with at most $\mathsf{denominator}(\alpha) + 2$ *states.*

For $\mathsf{denominator}(\alpha) \leq 6$, these transducer automata are shown in Figures 3–8. In some cases, these transducers could be simplified by merging equivalent states.

Proof of Theorem 3. We first consider the case that $\alpha = p/q$ is a rational number. If q is even, we consider the intervals

$$J_0 := \{0\}, J_1 := \left(0, \frac{1}{q}\right), J_2 := \left[\frac{1}{q}, \frac{2}{q}\right), \ldots, J_q := \left[\frac{q-1}{q}, 1\right),$$

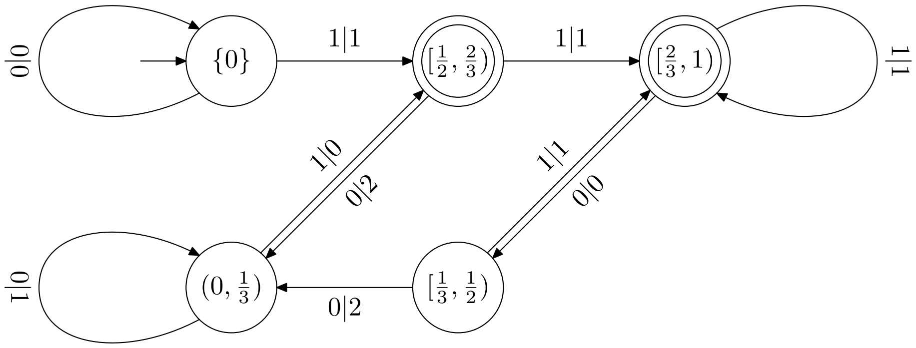

FIGURE 4. Right-To-Left-Transducer for $\alpha = 4/3$.

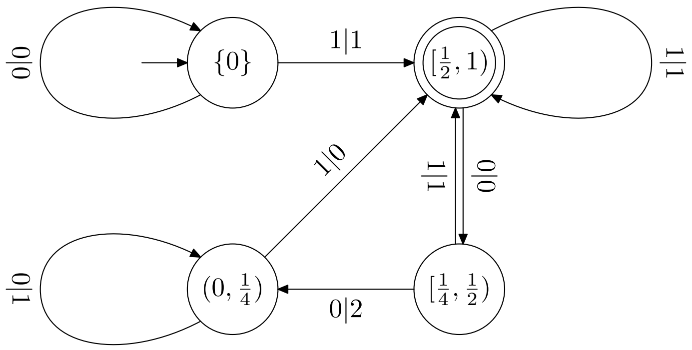

FIGURE 5. Right-To-Left-Transducer for $\alpha = 5/4$.

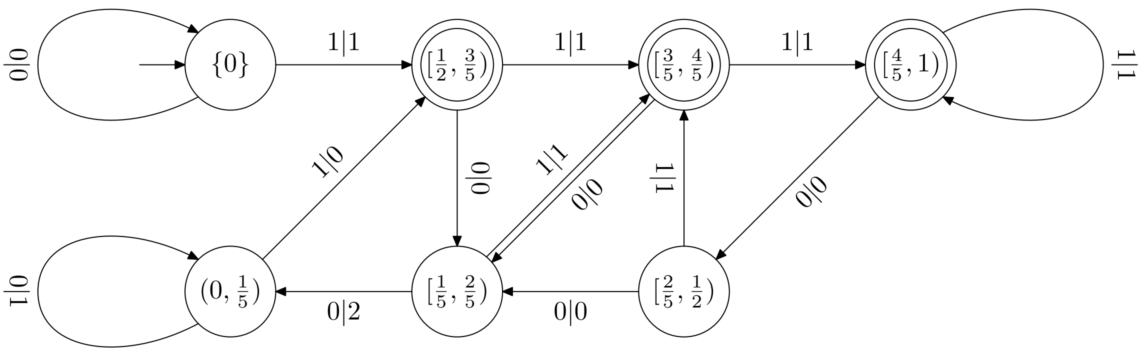

FIGURE 6. Right-To-Left-Transducer for $\alpha = 6/5$.

where J_1 is open and $J_2, \ldots, J_q$ are closed on the left and open on the right. If q is odd, we divide the middle interval and set

$$J_0 := \{0\}, J_1 := \left(0, \frac{1}{q}\right), J_2 := \left[\frac{1}{q}, \frac{2}{q}\right), \ldots, J_{\frac{q-1}{2}} := \left[\frac{q-3}{2q}, \frac{q-1}{2q}\right), J_{\frac{q+1}{2}} := \left[\frac{q-1}{2q}, \frac{1}{2}\right),$$
$$J_{\frac{q+3}{2}} := \left[\frac{1}{2}, \frac{q+1}{2q}\right), J_{\frac{q+5}{2}} := \left[\frac{q+1}{2q}, \frac{q+3}{2q}\right), \ldots, J_{q+1} := \left[\frac{q-1}{q}, 1\right).$$

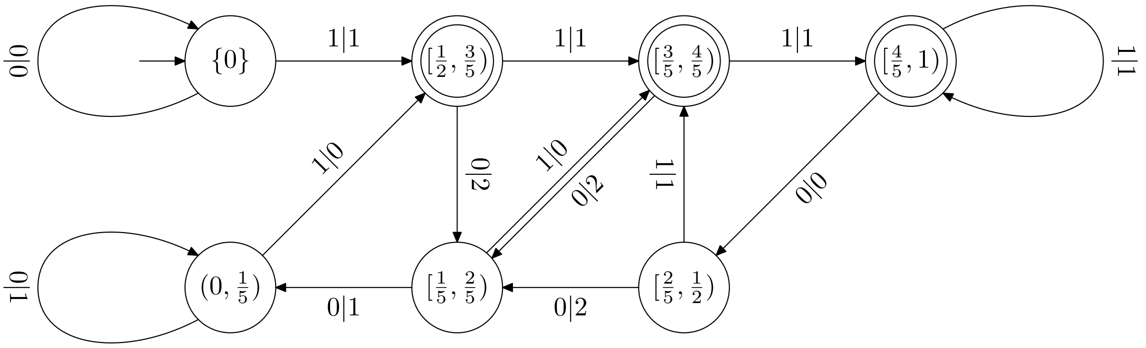

FIGURE 7. Right-To-Left-Transducer for $\alpha = 7/5$.

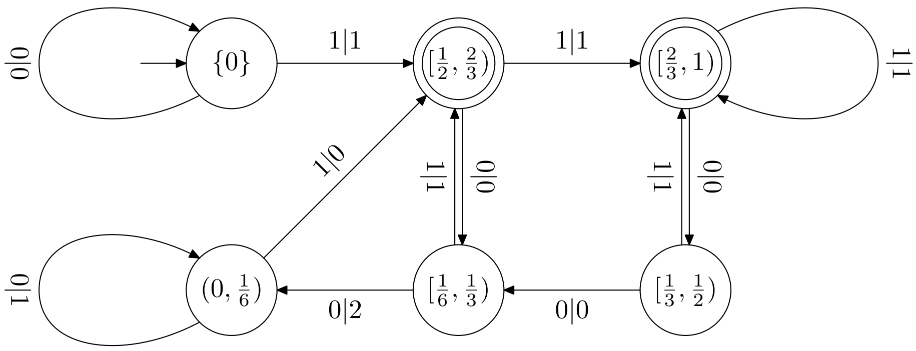

FIGURE 8. Right-To-Left-Transducer for $\alpha = 7/6$.

We consider the functions

$$f_d(x) := \frac{d}{2} + \frac{x}{2}, \qquad d = 0, 1,$$

and set

$$V := \{J_0, \ldots, J_{q+[q \text{ is odd}]}\}.$$

It is easily verified that for each $J_j \in V$ and $d \in \{0,1\}$ there is a unique $J_k \in V$ and a unique $o \in \{0,1,2\}$ such that $f_d(J_j) \subseteq J_k \cap I_o$, where the sets I_o have been defined in Theorem 1.

We define the transducer $\mathcal{T}$ by its set of states V and set of transitions

$$E := \{J_j \xrightarrow{d|o} J_k : J_j, J_k \in V, d \in \{0,1\}, o \in \{0,1,2\} \text{ such that } f_d(J_j) \subseteq J_k \cap I_o\}.$$

The initial state is $J_0 = \{0\}$, the terminal states are the states J_k with $J_k \subseteq [1/2, 1)$.

We claim that $\mathcal{T}$ is exactly the transducer we are looking for. Let n be a positive integer with standard binary expansion $(b_L, \ldots, b_0)$ satisfying $b_L = 1$. Assume that

$$\left\{\frac{n}{2^\ell}\right\} = \frac{\mathsf{value}(b_{\ell-1}, \ldots, b_0)}{2^\ell} \in J_j$$

for some $0 \le \ell < L$ and some state $J_j \in V$. Note that for $\ell = 0$, this state J_j is the initial state J_0. Now,

$$\left\{\frac{n}{2^{\ell+1}}\right\} = \frac{\mathsf{value}(b_\ell, b_{\ell-1}, \ldots, b_0)}{2^{\ell+1}} = f_{b_\ell}\left(\frac{\mathsf{value}(b_{\ell-1}, \ldots, b_0)}{2^\ell}\right) \in J_k \cap I_o$$

for the unique pair $(J_k, o) \in V \times \{0, 1, 2\}$ such that $J_j \xrightarrow{d|o} J_k$ is a transition in $\mathcal{T}$. By Theorem 1, the digit o is correct. By induction, we see that $\mathcal{T}$ is correct.

This completes the proof for rational α.

Conversely, we now assume that there is an appropriate transducer $\mathcal{T}$ with set of vertices $V = \{1, \ldots, n\}$ and set of transitions E. Our strategy is to count the number $S_2(2^L)$ of digits 2 in the expansions of the integers $\{1, \ldots, 2^{L-1}\}$ using the transducer and compare this with Theorem 2 to obtain a contradiction.

We consider the labelled transition matrix $A(Y)$ with entries

$$a_{jk} = \sum_{j \xrightarrow{d|o} k \in E} Y^{[o=2]}, \qquad 1 \le j, k \le n,$$

i.e., transitions with output label 2 are labelled with Y, all others contribute summands 1. Set $m_{K,L}$ to be the number of positive integers in the set $\{2^{L-1}, \ldots, 2^L - 1\}$ with the property that its α-expansion has exactly K occurrences of the digit 2. We study the generating function

$$G(Y, Z) := \sum_{\substack{K \ge 0 \\ L \ge 1}} m_{K,L} Y^K Z^L = v^t (I - AZ)^{-1} w,$$

where $v = (1, 0, \ldots, 0)^t$ and w is the vector with entries $[j \text{ is a terminal state}]$, $j = 1, \ldots, n$. Obviously, $G(Y, Z)$ is a rational function in Y and Z over $\mathbb{Q}$. Then the quantity $S_2(2^L) - S_2(2^{L-1})$ equals the coefficient of Z^L in $G_Y(1, Z)$, where G_Y denotes differentiation with respect to Y. It is clear that $G_Y(1, Z)$ is a rational function in Y over $\mathbb{Q}$. Since

$$S_2(2^L) - S_2(2^{L-1}) = \frac{\alpha - 1}{4} L 2^L + O(2^L)$$

by Theorem 2, we see that $1/2$ is a double pole of $G_Y(1, Z)$. We conclude that

$$\frac{\alpha - 1}{4} = \lim_{Z \to 1/2} G_Y(1, Z)(2Z - 1)^2 \in \mathbb{Q},$$

which is a contradiction to the irrationality of α. □

Remark 4.1. From the transducers that we have constructed for rational α, it can be concluded that the *set of admissible representations*, i.e., those words over the alphabet $\{0, 1, 2\}$, which occur as α-greedy representation for some natural number n, is a *regular* *set.* Here are a few examples:

$$\begin{aligned}
\alpha = 3/2: &\qquad (1 + 01^*2)^*10^* \\
\alpha = 5/4: &\qquad (1 + 10 + 01^*20)^*10^* \\
\alpha = 4/3: &\qquad (\varepsilon + (10+1)^*1)(01^*20(10+1)^*1 + 01^*2)^*10^*
\end{aligned}$$

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