Subblock Occurrences in Positional Number Systems and Gray code Representation

Peter Kirschenhofer
Helmut Prodinger
Institut fur Algebra und Diskrete Mathematik
Technische Universität Wien
Gußhausstraße 27-29, A-1040 Wien
Austria

ABSTRACT

This paper deals with the average number of subblock occurrences in the following representations of integers : the $<q, d>$ −ary representation (with digits d, $d+1, \dots, d+q-1$) and the Gray code representation.

1. INTRODUCTION

In a recent paper [3] P. Kirschenhofer has proved the following result on the number $B_q(w, n)$ of subblocks w in the q-ary representation of $n \in \mathbf{N}_0$, where overlapping is allowed and w is a string of digits of length s neither starting nor ending with 0 :

$$(1) \quad \frac{1}{m}\sum_{n=0}^{m-1} B_q(w, m) = \frac{\log_q m-(s-1)}{q^s} + H_w(\log_q m) + \frac{E_w(m)}{m},$$

where H_w is continuous, periodic with period 1 and $H_w(0)=0$ and E_w is bounded.

For the special case $q=2$ and $w=1^s$ this result has already appeared in [5].

The method to establish this result is an approach to apply a method which has been introduced by Delange in [1] where he has analyzed the sum of digits function.

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(*Theorem 1*) – the Fourier expansion of a periodic function appearing in this average constitutes the next result (*Theorem 2*). At the end of the section we use the previous results to establish the desired average

$$\frac{1}{m}\sum_{n=0}^{m-1} B_{q,d}(w, n)$$

(*Lemma 2*, resp. *Theorem 3*).

LEMMA 1. *Let w be a sequence of s digits not starting with O and let $A_{q,d}(w, r)$ denote the number of subblocks w in the $<q, d>$-representation of the real $r \geqslant 0$, where we count all those occurrences that are either entirely to the left or straddle the radix point. Then*

$$A_{q,d}(w, r) = \sum_{k\geqslant 1}\left(\left\lfloor \frac{r}{q^k} + \beta + \frac{1}{q^s}\right\rfloor - \left\lfloor \frac{r}{q^k} + \beta\right\rfloor\right), \tag{2}$$

where

$$\beta = 1 - (o\,.\,w)_{q,d} - \frac{1}{q^s}\cdot\frac{d}{q-1} - \frac{1}{q^s}. \tag{3}$$

PROOF. The k-term in the sum of the Lemma can only take the values 0 and 1. In the following we will show that it takes the value 1 iff w occurs as a subblock in the $<q, d>$-representation of r (starting with the k-th digit left to the radix point).

We define the number ε by the equation

$$q^{k-s}\,.\,\varepsilon = r - q^k\,(o\,.\,w)_{q,d} - q^k\left\lfloor \frac{r}{q^k} - \frac{d}{q-1}\right\rfloor.$$

(The last term corresponds to the digits left of the k-th position.)

Then it follows that

$$\frac{d\,.\,q^s}{q-1} - (o\,.\,w)_{q,d}\,.\,q^s \leqslant \epsilon < \frac{dq^s}{q-1} - (o\,.\,w)_{q,d}\,.\,q^s + q^s$$

and it can be readily checked that a subblock w starting at k-th position corresponds to values ε in the interval

$$\frac{d}{q-1} \leqslant \epsilon < \frac{d}{q-1} + 1.$$

Indeed, the k-term of the sum is 1 in this case. To make the discussion of the remaining cases for ϵ independent from the special form of w, we observe that

$$\frac{d}{q-1}\left(1-\frac{1}{q^s}\right) \leqslant (o\,.\,w)_{q,d} \leqslant \frac{d}{q-1}\left(1-\frac{1}{q^s}\right)+1-\frac{1}{q^s}.$$

The remaining intervals for ϵ are covered by

$$\frac{d}{q-1}-q^s+1 \leqslant \epsilon < \frac{d}{q-1}$$

and

$$\frac{d}{q-1}+1 \leqslant \epsilon < \frac{d}{q-1}+q^s.$$

In both cases the k-term of the sum is 0. □

For later use we remark that the proof of Lemma 1 contains also the following corollary.

COROLLARY 1. *The number $B_{q,d}(w, r)$ of subblocks w in the situation of Lemma 1 that are entirely to the left of the radix point is given by*

$$(4) \qquad B_{q,d}(w, r) = \sum_{k \geqslant s}\left(\left\lfloor \frac{r}{q^k}+\beta+\frac{1}{q^s}\right\rfloor - \left\lfloor \frac{r}{q^k}+\beta\right\rfloor\right).$$

Following the plan indicated at the beginning of this section we turn now to the investigation of the average of $B_{q,d}(w, n)$:

In a first step we compute the average $A_{q,d}(w, r)$.

THEOREM 1. *With w and $A_{q,d}(w, r)$ as in Lemma 1 we have*

$$(5) \qquad \frac{1}{m}\int_0^m A_{q,d}(w, r)\,dr = \frac{\log_q m}{q^s} + H_w(\log_q m)$$

where H_w is a continuous, periodic function with period 1 and $H_w(0)=0$.

PROOF. With the explicit formula for $A_{q,d}(w, r)$ of Lemma 1 we get

$$\int_0^m A_{q,d}(w, r)\,dr = \int_0^m \sum_{k \geqslant 1}\left(\left\lfloor \frac{r}{q^k}+\beta+\frac{1}{q^s}\right\rfloor - \left\lfloor \frac{r}{q^k}+\beta\right\rfloor\right) dr.$$

We observe that nonzero contributions to the sum may originate only from values of $k \leqslant l+1$ where

$$l = \left\lfloor \log_q m - \log_q\left(1+\frac{d}{q-1}\right)\right\rfloor$$

so that

$$(6)\quad \int_0^m A_{q,d}(w,r)\,dr = \sum_{k=1}^{l+1} \int_0^m \left(\left\lfloor \frac{r}{q^k}+\beta+\frac{1}{q^s}\right\rfloor - \left\lfloor \frac{r}{q^k}+\beta\right\rfloor\right) dr.$$

It is convenient to introduce the function

$$(7)\quad g_{\beta,s}(x) = \int_0^x \left(\left\lfloor u+\beta+\frac{1}{q^s}\right\rfloor - \left\lfloor u+\beta\right\rfloor - \frac{1}{q^s}\right) du.$$

Then $g_{\beta,s}$ is continuous, periodic with period 1, $g_{\beta,s}(0)=0$, and a simple substitution (compare with [3]) shows that the sum from above equals

$$\frac{1}{q^s}\, m(l+1) + \sum_{k\geqslant 0} q^{l+1-k}\; g_{\beta,s}(mq^{k-l-1}).$$

With $\{x\}=x-\lfloor x\rfloor$ and $\gamma=\log_q\left(1+\dfrac{d}{q-1}\right)$ we can rewrite (6) as

$$\frac{1}{q^s}\cdot m\cdot \log_q m + \frac{m}{q^s}\,(1-\gamma-\{-\gamma+\log_q m\})$$

$$+ mq^{1-\gamma-\{-\gamma+\log_q m\}}\cdot h_{\beta,s}(q^{-1+\gamma+\{-\gamma+\log_q m\}}),$$

where

$$(8)\quad h_{\beta,s}(x) = \sum_{k\geqslant 0} q^{-k}\cdot g_{\beta,s}(xq^k).$$

Putting

$$(9)\quad H_w(x) = \frac{1-\gamma-\{-\gamma+x\}}{q^s} + q^{1-\gamma-\{-\gamma+x\}}\cdot h_{\beta,s}(q^{\{-\gamma+x\}-1+\gamma}),$$

the function $H_w(x)$ is continuous, periodic with period 1 and $H_w(0)=0$, and Theorem 1 is established. □

It is instructive to compute the Fourier coefficients of the periodic function appearing in Theorem 1.

THEOREM 2. *The periodic function $H_w(x)$ of Theorem 1 has the following Fourier expansion* :

$$H_w(x)=\sum_{k\in\mathbf{Z}} h_k . e^{2\pi ikx} \tag{10}$$

with

$$h_0=\log_q \frac{\Gamma(1-\{\beta+q^{-s}\})}{\Gamma(1-\{\beta\})}-\frac{1}{q^s}\left(\frac{1}{2}+\frac{1}{\log q}\right),$$

$$h_k=\frac{\zeta(\chi_k, 1-\{\beta+q^{-s}\})-\zeta(\chi_k, 1-\{\beta\})}{\log q \,.\, \chi_k \,.\, (1+\chi_k)}, \; k\neq 0 \tag{11}$$

where $\{x\}=x-\lfloor x\rfloor$ denotes the fractional part of x, ζ, (z, a) the ζ-function of Hurwitz, $\chi_k=2k\pi i/\log q$ and β is defined as in Lemma 1.

PROOF. Let $\gamma=\log_q\left(1+\frac{d}{q-1}\right)$ and assume $\gamma\leqslant x<\gamma+1$. Then

$$H_w(x)=\frac{1-x}{q^s}+q^{1-x} \,.\, h_{\beta,s}(q^{x-1})$$

and

$$h_k=a_k+b_k$$

with

$$a_k=\int_\gamma^{\gamma+1} q^{1-x}\, h_{\beta,s}(q^{x-1})\, e^{-2k\pi ix} dx,$$

$$b_k=\frac{1}{q^s}\int_\gamma^{\gamma+1}(1-x)\, e^{-2k\pi ix} dx.$$

It is readily verified that

$$b_0=\frac{1}{q^s}\left(\frac{1}{2}-\gamma\right),$$

$$b_k=\frac{1}{q^s} \,.\, \frac{e^{-2k\pi i\gamma}}{2k\pi i}, \; k\neq 0.$$

Further $$a_k = \sum_{r=0}^{\infty} \int_{\gamma}^{\gamma+1} q^{1-x-r} \cdot g_{\beta,s}(q^{x-1+r}) e^{-2k\pi i x} dx.$$

Using the substitution $x = 1 - r + \log_q u$ we get

$$a_k = \frac{1}{\log q} \cdot \int_{q^{\gamma-1}}^{\infty} \frac{g_{\beta,s}(u)}{u^{2+\chi_k}} du.$$

With the abbreviation

(12) $$\Phi_{\beta,s}(z) = \int_{q^{\gamma-1}}^{\infty} \frac{g_{\beta,s}(u)}{u^{z+1}} du$$

we may write

$$a_k = \frac{1}{\log q} \cdot \Phi_{\beta,s}(1+\chi_k).$$

In the following, we compute $\Phi_{\beta,s}(z)$ for $z \neq 1$:

$$\Phi_{\beta,s}(z) = \int_{q^{\gamma-1}}^{\infty} \frac{du}{u^{z+1}} \int_0^u \left(\left\lfloor t+\beta+\frac{1}{q^s} \right\rfloor - \left\lfloor t+\beta \right\rfloor - \frac{1}{q^s} \right) dt$$

$$= -\frac{1}{zq^{(\gamma-1)z+s+1}} \left(1+\frac{d}{q-1} \right) + I_1 - I_2 - I_3$$

with

$$I_1 = \frac{1}{z} \int_{q^{\gamma-1}}^{\infty} \frac{du}{u^z} \left\lfloor u+\beta+\frac{1}{q^s} \right\rfloor du,$$

$$I_2 = \frac{1}{z} \int_{q^{\gamma-1}}^{\infty} \frac{du}{u^z} \lfloor u+\beta \rfloor du,$$

$$I_3 = \frac{1}{zq^s} \int_{q^{\gamma-1}}^{\infty} \frac{du}{u^z} = \frac{1}{z(z-1)q^{(\gamma-1)(z-1)+s}}.$$

Evaluating I_1 and I_2 we derive

$$I_1 = J_1 + J_2 + \frac{1}{z(z-1)} \cdot \zeta(z-1,\ 1-\{\beta+q^{-s}\})$$

with
$$J_1 = \frac{1}{z}\left\lfloor \beta + \frac{1}{q^s} \right\rfloor \int_{q^{\gamma}-1}^{\infty} \frac{du}{u^z},$$

$$J_2 = \frac{1}{z} \int_{q^{\gamma}-1}^{1-\{\beta+q^{-s}\}} \frac{\lfloor u+\{\beta+q^{-s}\}\rfloor}{u^z}\, du$$

and
$$I_2 = J_3 + J_4 + \frac{1}{z(z-1)} \cdot \zeta(z-1,\ 1-\{\beta\})$$

with
$$J_3 = \frac{1}{z} \lfloor\beta\rfloor \int_{q^{\gamma}-1}^{\infty} \frac{du}{u^z},$$

$$J_4 = \frac{1}{z} \int_{q^{\gamma}-1}^{\infty} \frac{\lfloor u+\{\beta\}\rfloor}{u^z}\, du.$$

By the proof of Lemma 1, $\lfloor\beta+q^{-s}\rfloor - \lfloor\beta\rfloor = 0$, so that

$$J_1 - J_3 = 0.$$

Furthermore, we show that $J_2 = J_4 = 0$, since

$$0 < q^{\gamma-1} + \left\{ \beta + \frac{1}{q^s} \right\} \leqslant 1$$

and
$$0 < q^{\gamma-1} + \{\beta\} \leqslant 1 :$$

The first inequality follows by

$$0 < q^{\gamma-1} + \left\{ \beta + \frac{1}{q^s} \right\}$$

$$= \frac{1}{q} + \frac{d}{q-1}\left(\frac{1}{q} - \frac{1}{q^s} \right) + \{-(0\,.\,w)_{q,d}\} = A.$$

Let $(0\,.\,w)_{q,d} = \sum_{i=1}^{s} w_i\, q^{-i}$ (observe $w_1 \neq 0$!).

In the case $w_1 < 0$ we have

$$A \leqslant \frac{1-w_1}{q} - \sum_{i=2}^{s} \frac{w_i}{q^i}$$

$$< \frac{1-d}{q} + \frac{1}{q} = \frac{2-d}{q} \leqslant 1.$$

For $w_1 > 0$ we have

$$A = \frac{1}{q} + 1 - \frac{w_1}{q} - \sum_{i=2}^{s} \frac{w_i}{q^i} + \frac{d}{q-1}\left(\frac{1}{q} - \frac{1}{q^s}\right)$$

$$\leqslant 1 - \frac{d}{q(q-1)}\left(1 - \frac{1}{q^{s-1}}\right) + \frac{d}{q-1}\left(\frac{1}{q} - \frac{1}{q^s}\right) = 1.$$

Since
$$\{\beta\} = \left\{\beta + \frac{1}{q^s}\right\} - \frac{1}{q^s}$$

the same arguments work for the proof of the second inequality.

Inserting $z = 1 + \chi_k$ into $\Phi_{\beta,s}(z)$ we get

$$(13) \quad a_k = \frac{1}{2k\pi i} \cdot \left(\frac{1}{1+\chi_k}\left(\zeta(\chi_k, 1-\{\beta+q^{-s}\}) - \zeta(\chi_k, 1-\{\beta\})\right) - \frac{1}{q(\gamma-1)\chi_k + s}\right) \text{ for } k \neq 0,$$

and the formula for h_k $(k \neq 0)$ is established.

In the instance $k=0$, we need

$$\lim_{z \to 1} \Phi_{\beta,s}(z).$$

Using the expansion of $\zeta(z, a)$ about $z=1$ (compare with [6]) this limit turns out to be

$$-\frac{1}{q^s}(1-\gamma)\log q + \log \Gamma\left(1 - \left\{\beta + \frac{1}{q^s}\right\}\right) - \log \Gamma(1-\{\beta\}) - \frac{1}{q^s},$$

and a_0 and thus h_0 is computed immediately. □

We continue our investigation by studying the average of the number of subblocks w that straddle the radix point :

LEMMA 2. *With w, $A_{q,d}(w, r)$ and $B_{q,d}(w, r)$ as in Lemma 1 resp. Corr. 1 we have*

$$(14)\qquad \frac{1}{m}\int_0^m (A_{q,d}(w, r)-B_{q,d}(w, r))dr=\frac{(s-1)}{q^s}-\frac{E_w(m)}{m}$$

where $E_w(m)$ is bounded.

PROOF. By Corollary 1

$$\int_0^m (A_{q,d}(w, r)-B_{q,d}(w, r))\, dr$$

$$=\sum_{k=1}^{s-1}\int_0^m\left(\left\lfloor \frac{r}{q^k}+\beta+\frac{1}{q^s}\right\rfloor-\left\lfloor \frac{r}{q^k}+\beta\right\rfloor\right)dr$$

$$=m\cdot\frac{s-1}{q^s}+\sum_{k=1}^{s-1} q^k g_{\beta,s}\left(\frac{m}{q^k}\right)=m\frac{s-1}{q^s}-E_w(m)$$

(with $g_{\beta,s}(x)$ from (7)). □

Combining Theorem 1 and Lemma 2 yields an expression for

$$\int_0^m B_{q,d}(w, r)\, dr.$$

This integral equals the "truncated sum"

$$(15)\qquad \frac{q-1+d}{q-1}\cdot B_{q,d}(w, 0)+B_{q,d}(w, 1)+\ldots+B_{q,d}(w, m-1)+\frac{-d}{q-1}B_{q,d}(w, m)$$

rather than the desired sum

$$B_{q,d}(w, 0)+\ldots+B_{q,d}(w, m-1).$$

THEOREM 3. *Let $B_{q,d}(w, n)$ denote the number of subblocks w in the $<q, d>$-representation of $n\in N_0$, where w is a sequence of s digits not* *starting with 0. Then*

$$\frac{1}{m}\sum_{n=0}^{m-1} B_{q,d}(w, n) = \frac{1}{m}\int_0^m B_{q,d}(w, r)\,dr + \frac{d}{q-1} \cdot \frac{B_{q,d}(w, m)}{m}$$

(16)

$$= \frac{\log_q m}{q^s} + H_w(\log_q m) - \frac{s-1}{q^s} + \frac{E_w(m)}{m} + \frac{d}{q-1} \cdot \frac{B_{q,d}(w, m)}{m}$$

where H_w is the periodic function analyzed in Theorem 2 and $E_w(m)$ is bounded. Obviously, $B_{q,d}(w, m)=0(\log m)$.

3. FURTHER ANALYSIS OF THE ERROR TERM $E_w(m)$

It is not difficult to see that $E_w(m)$ oscillates in a rather irregular way. In order to get information about $E_w(m)$ we study its *average value*

$$\frac{1}{m}\sum_{n=0}^{m-1} E_w(n)$$

and prove the following

THEOREM 4. *Let $w=w_1\ldots w_r\,0^{s-r}$ with $w_1, w_r \neq 0$ and β be defined as in Lemma 1. Then*

$$(17)\quad \frac{1}{m}\sum_{n=0}^{m-1} E_w(n) = \frac{q^s-q}{2q^s(q-1)} - \frac{s-1}{2q^s} - \frac{1}{q^s}\sum_{k=0}^{s-1} \lfloor q^k\{\beta\}\rfloor$$

$$+\sum_{k=r}^{s-1} \frac{2-\{q^k\beta\}}{q^k} + 0\left(\frac{1}{m}\right),\ m\to\infty.$$

PROOF. We have

$$\frac{1}{m}\sum_{n=0}^{m-1} E_w(n) = -\frac{1}{m}\sum_{n=0}^{m-1}\sum_{k=0}^{s-1} q^k \cdot g_{\beta,s}\left(\frac{n}{q^k}\right)$$

$$= -\frac{1}{m}\sum_{n=0}^{m-1}\sum_{k=0}^{s-1} q^k \cdot g_{\beta,s}\left(\left\{\frac{n}{q^k}\right\}\right)$$

$$=-\frac{1}{m}\sum_{k=0}^{s-1}\left\lfloor\frac{m-1}{q^k}\right\rfloor\sum_{n=0}^{q^k-1}q^k\,g_{\beta,s}\left(\frac{n}{q^k}\right)$$

$$=-\sum_{k=0}^{s-1}\sigma_k+0\left(\frac{1}{m}\right)$$

where

$$\sigma_k=\sum_{n=1}^{q^k}q^{-k}\int_0^n\left(\left\lfloor\frac{t}{q^k}+\beta+\frac{1}{q^s}\right\rfloor-\left\lfloor\frac{t}{q^k}+\beta\right\rfloor-\frac{1}{q^s}\right)dt.$$

Now we have for $\delta\in\mathbf{R}$

$$\int_0^n\left\lfloor\frac{t}{q^k}+\delta\right\rfloor=n\lfloor\delta\rfloor+\sum_{j=0}^{n-1}\left\lfloor\frac{j}{q^k}+\{\delta\}\right\rfloor+\{q^k\delta\}\left(\left\lfloor\frac{n}{q^k}+\delta\right\rfloor-\lfloor\delta\rfloor\right)$$

and therefore, observing

$$\left\lfloor\beta+\frac{1}{q^s}\right\rfloor-\lfloor\beta\rfloor=0$$

respectively

$$\left\{\beta+\frac{1}{q^s}\right\}=\{\beta\}+\frac{1}{q^s}$$

a lengthy but elementary computation yields

$$\sigma_k=\frac{\left(\{q^k\beta\}-\left\{q^k\left(\beta+\frac{1}{q^s}\right)\right\}\right)\left(\{q^k\beta\}+\left\{q^k\left(\beta+\frac{1}{q^s}\right)\right\}-3\right)}{2q^k}+$$

$$+q^{k-s}\left(\{\beta\}-\frac{1}{2}\right)+\frac{1}{2}q^{k-2s}-q^{-s}.$$

In the following we use that

$$\left\{q^k\left(\beta+\frac{1}{q^s}\right)\right\}-\{q^k\beta\}=\begin{cases}q^{k-s} & \text{for } 0\leqslant k\leqslant r-1\\ q^{k-s}-1 & \text{for } r\leqslant k\leqslant s-1\end{cases}$$

which may be verified along the lines of the proof of Lemma 1, Thus we can rewrite σ_k as

$$\frac{q^k\{\beta\}-\{q^k\beta\}}{q^s}-\frac{1}{2}q^{k-s}+\frac{1}{2}q^{-s}+\begin{cases} 0 & \text{for } 0\leqslant k\leqslant r-1 \\ \dfrac{\{q^k\beta\}-2}{q^k} & \text{for } r\leqslant k\leqslant s-1 \end{cases}$$

which completes the proof of the theorem. □

4. SUBBLOCK OCCURRENCES IN GRAY CODE REPRESENTATION

A *Gray Code* is an encoding of the integers as sequences of bits with the property that the representations of adjacent integers differ in exactly one binary position. As in [2], we restrict our considerations to the *standard Gray (or binary reflected)* Code. The following table shows the Gray Code representations of the first 16 nonnegative natural numbers (n refers to the number and k to the position).

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
1			1	1	1	1	0	0	0	0	1	1	1	1	0	0
2				1	1	1	1	1	1	1	1	1	0	0	0	0
3									1	1	1	1	1	1	1	1

In the k-th row we find the pattern $0^{2^k}1^{2^{k+1}}0^{2^{k+1}}1^{2^{k+1}}0^{2^{k+1}}\ldots$.

If we know how to count occurrences of subblocks in binary, then we can count occurrences of subblocks in Gray Code quite easily :

Then k-th bit in the Gray Code representation of an integer is simply the exclusive OR of the k-th and $(k+1)$-st bits in the binary representation of the same integer. Regarding that the exclusive OR is simply representable as addition mod 2 ($\oplus$) of the concerned bits, the subblock $w=w_1\ldots w_s$ in Gray Code corresponds to one of the two patterns

$u(w)=u_1 \dots u_{s+1}$ resp. $v(w)=v_1 \dots v_{s+1}$ where $u_i = \bigoplus_{j=1}^{i-1} w_j$ and $v_i = u_i \oplus 1$.

Thus we obtain as a corollary of Theorem 3 :

THEOREM 5. *Let* $B_{GC}(w, n)$ *denote the number of subblocks* w *in the Gray Code representation of* $n \in \mathbf{N}_0$, *where* w *is a sequence of* s *digits not starting with zero and let* $u(w)$ *resp.* $v(w)$ *be defined as above. Then*

$$\frac{1}{m}\sum_{n=0}^{m-1} B_{GC}(w, n) = \tag{18}$$

$$\frac{\log_2 m}{2^s} + H_{u(w)}(\log_2 m) + H_{v(w)}(\log_2 m) - \frac{s}{2^s} + 0\left(\frac{1}{m}\right),$$

where $H_{u(w)}$ resp. $H_{v(w)}$ *are the periodic functions (for the instance* $q=2$, $d=0$) *which are analyzed in Theorem 2.*

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