ON THE ORDER OF THE RECURSION RELATION OF MOTZKIN NUMBERS OF HIGHER RANK

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Abstract. For Motzkin paths with up- and down-steps of heights 1 and 2, the minimal recursion is of order 6, not of order 4, as conjectured by Schork.

We consider higher rank Motzkin numbers, as suggested by Schork [2]: There are up-steps $(1,1)$, $(1,2)$, ..., $(1,r)$ with respective weights $a_1,\dots,a_r$, down-steps $(1,-1)$, $(1,-2)$, ..., $(1,-r)$ with respective weights $c_1,\dots,c_r$, and a level-step $(1,0)$ with weight b.

Let us first consider the classical case $r=1$. The generating function $M(z)$ of these paths satisfies the equation

$$M = 1 + bzM + azMczM,$$

whence

$$\frac{1-bz-\sqrt{1-2bz+b^2z^2-4az^2c}}{2az^2c}.$$

Schork's first problem is to find a recursion for the numbers $m_n = [z^n]M(z)$.

This can be automatically solved with Maple's program gfun (written by Salvy et al.): The procedure algeqtodiffeq translates the (algebraic) equation for $M(z)$ into an equivalent differential equation:

$$2 + (3bz - b^2z^2 + 4az^2c - 2)M + (-z + 2bz^2 - z^3b^2 + 4z^3ac)M'.$$

The procedure diffeqtorec translates the differential equation into a recursion:

$$(-b^2+4ac)(n+1)m_n + (5b+2bn)m_{n+1} - (n+4)m_{n+2} = 0,$$

which solves already this first problem.[1]

Now let us move to the instance $r=2$. Let us assume that the weights are all 1. In the paper [1] we find the equation for the generating function:

$$z^4M^4 - z^2(1+z)M^3 + z(2+z)M^2 - (1+z)M + 1 = 0.$$

Thus (again with gfun)

$$\begin{aligned}
&-4 - 100z^2 + 56z + (3750z^6 - 5000z^5 + 250z^4 + 700z^3 + 160z^2 - 92z + 4)M \\
&\quad + (-328z^2 + 32z - 15250z^6 - 20z^3 + 4750z^4 + 11250z^7 - 650z^5)M' \\
&\quad + (5625z^8 - 7750z^7 - 1200z^6 + 3880z^5 - 395z^4 - 186z^3 + 26z^2)M'' \\
&\qquad + (625z^9 - 875z^8 - 250z^7 + 610z^6 - 91z^5 - 23z^4 + 4z^3)M''' = 0
\end{aligned}$$

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[1]After sending a draft of this note to M. Schork, he informed me that he could now also establish this recurrence together with Mansour and Sun.

and

$$
\begin{aligned}
&625(n+3)(n+2)(n+1)m_n - 125(n+3)(n+2)(7n+27)m_{n+1}\\
&- 50(n+3)(5n^2+24n+23)m_{n+2} + (41890+30860n+7540n^2+610n^3)m_{n+3}\\
&+ (-6844-5151n-1214n^2-91n^3)m_{n+4} - (n+7)(23n^2+301n+976)m_{n+5}\\
&+ 2(2n+13)(n+8)(n+7)m_{n+6} = 0.
\end{aligned}
$$

(This recursion also appears in [1].)

Bruno Salvy has kindly informed me that this recursion of order 6 is *minimal*.

Schork [2] conjectured that there should be a $(2r+1)$-term recursion (=order $2r$). Thus, the conjecture does not hold.

References

[1] C. Banderier and P. Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002. Selected papers in honour of Maurice Nivat.

[2] M. Schork. On the recursion relation of Motzkin numbers of higher rank. *Online Journal of Analytic Combinatorics*, 2:#3, 2007.

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