DETERMINANTS CONTAINING RISING POWERS OF FIBONACCI NUMBERS

HELMUT PRODINGER

ABSTRACT. A matrix containing rising powers of Fibonacci numbers is investigated. The LU-decomposition is guessed and proved; this leads to a formula for the determinant. Similar results are also obtained for a matrix of Lucas numbers.

1. INTRODUCTION

Carlitz [1], motivated by earlier writings, loc. cit., computed the determinant

$$\begin{vmatrix} F_n^r & F_{n+1}^r & F_{n+2}^r & \cdots & \\ F_{n+1}^r & F_{n+2}^r & F_{n+3}^r & \cdots & \\ F_{n+2}^r & F_{n+3}^r & F_{n+4}^r & \cdots & \\ \cdots & \cdots & \cdots & \ddots & \\ & & & & F_{n+r}^r \end{vmatrix},$$

with the result

$$(-1)^{\binom{r+1}{2}(n+1)}\prod_{j=0}^{r}\binom{r}{j}\cdot(F_1^rF_2^{r-1}\dots F_r)^2;$$

F_i are Fibonacci numbers as usual.

In the present note we consider the rising powers analogue

$$M=\begin{pmatrix} F_n^{\langle r\rangle} & F_{n+1}^{\langle r\rangle} & F_{n+2}^{\langle r\rangle} & \cdots & \\ F_{n+1}^{\langle r\rangle} & F_{n+2}^{\langle r\rangle} & F_{n+3}^{\langle r\rangle} & \cdots & \\ F_{n+2}^{\langle r\rangle} & F_{n+3}^{\langle r\rangle} & F_{n+4}^{\langle r\rangle} & \cdots & \\ \cdots & \cdots & \cdots & \ddots & \\ & & & & F_{n+r}^{\langle r\rangle} \end{pmatrix}.$$

This is an $(r+1)\times(r+1)$ matrix, and we assume that the indices run from $0,\dots,r$. The rising products are defined as follows:

$$F_m^{\langle r\rangle}:=F_mF_{m+1}\dots F_{m+r-1}.$$

Although this definition looks more complicated than the one used by Carlitz, it is actually nicer, since we are able to compute (first guessing, then proving) the LU-decomposition of $M=LU$, from which the determinant is an easy corollary, via $\det(M)=U_{0,0}U_{1,1}\dots U_{r,r}$.

2010 *Mathematics Subject Classification.* 11B39; 11C20; 15B36.

Key words and phrases. Fibonacci numbers; LU-decomposition; q-analogues; determinants.

The author was supported by an incentive grant of the National Research Foundation of South Africa.

2. The LU-decomposition of M

We start from the Binet form

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}=\alpha^{n-1}\frac{1-q^n}{1-q},$$

with

$$\alpha=\frac{1+\sqrt{5}}{2},\quad \beta=\frac{1-\sqrt{5}}{2},\quad q=\frac{\beta}{\alpha}=-\frac{1}{\alpha^2},$$

so that $\alpha=\mathrm{i}q^{-1/2}$. We write further

$$F_{n+j}=y\alpha^{j-1}\frac{1-xq^j}{1-q},$$

with

$$y=\alpha^n\quad\text{and}\quad x=q^n.$$

Thus

$$M_{i,j}=F_{n+i+j}^{\langle r\rangle}=\frac{y^r}{(1-q)^r}\alpha^{(i+j-1)r+\binom{r}{2}}(xq^{i+j};q)_r.$$

Here, we use standard q-notation: $(x;q)_m:=(1-x)(1-xq)\dots(1-q^{m-1})$.

This is the form that we use to guess (and then prove) the LU-decomposition. It holds for general variables x, y, q, α. However, for our application, we will then specialize. For these specializations, we need the notation of a Fibonacci-factorial:

$$n!_F:=F_1F_2\dots F_n.$$

Theorem 1. *For $0\le i\le j\le r$,*

$$U_{i,j}=\frac{x^iy^r}{(1-q)^r}\alpha^{r(i+j)+\frac{r(r-3)}{2}}q^{\frac{3(i-1)i}{2}}(-1)^i\frac{(x;q)_{j+r}(x;q)_{i-1}(q;q)_j(q;q)_r}{(x;q)_{i+j}(x;q)_{2i-1}(q;q)_{r-i}(q;q)_{j-i}}.$$

For $0\le j\le i\le r$,

$$L_{i,j}=\frac{(x;q)_{i+r}(q;q)_i(x;q)_{2j}}{(x;q)_{j+r}(x;q)_{i+j}(q;q)_j(q;q)_{i-j}}\alpha^{r(i-j)}.$$

Corollary 1. *The specialized versions (Fibonacci numbers) are as follows:*

$$U_{i,j}=\frac{(n+j+r-1)!_F\,(n+i-2)!_F\,j!_F\,r!_F}{(n+i+j-1)!_F\,(n+2i-2)!_F\,(r-i)!_F\,(j-i)!_F}(-1)^{\frac{i(i+1)}{2}+ni},$$

$$L_{i,j}=\frac{(n+i+r-1)!_F\,(n+2j-1)!_F\,i!_F}{(n+j+r-1)!_F\,(n+i+j-1)!_F\,j!_F\,(i-j)!_F}.$$

Theorem 2. *The determinant of the matrix M is given by*

$$\det(M)=\prod_{i=0}^{r}U_{i,i}=(-1)^{\binom{r+2}{3}+n\binom{r+1}{2}}(r!_F)^{r+1}\prod_{i=0}^{r}\frac{(n+i+r-1)!_F\,(n+i-2)!_F}{(n+2i-1)!_F\,(n+2i-2)!_F}$$

$$=(-1)^{\binom{r+2}{3}+n\binom{r+1}{2}}(r!_F)^{r+1}.\quad\square$$

Although it is not necessary for our determinant calculation, we briefly mention two additional results (first general, then specialized):

Theorem 3.

$$U^{-1}_{i,j} = \frac{(q;q)_{2j}(q;q)_{r-j}(x;q)_{i+j-1}}{(q;q)_i(q;q)_r(q;q)_{j-i}(x;q)_{j-1}(x;q)_{i+r}} \times q^{-j(j-1)-ij+\frac{(i+1)i}{2}}(-1)^i(1-q)^r\alpha^{-r(i+j)-\frac{r(r-3)}{2}}x^{-j}y^{-r},$$

$$L^{-1}_{i,j} = \frac{(x;q)_{i+r}(x;q)_{i+j-1}(q;q)_i}{(x;q)_{j+r}(x;q)_{2i-1}(q;q)_j(q;q)_{i-j}} q^{\frac{i(i-1)}{2}-ij+\frac{(j+1)j}{2}}\alpha^{r(i-j)}(-1)^{i-j},$$

$$U^{-1}_{i,j} = \frac{(n+2j-1)!_F\,(n+i+j-2)!_F\,(r-j)!_F}{(n+j-2)!_F\,(n+i+r-1)!_F\,r!_F\,(j-i)!_F\,i!_F}(-1)^{ij+\frac{i(i-1)}{2}+rj},$$

$$L^{-1}_{i,j} = \frac{(n+i+r-1)!_F\,(n+i+j-2)!_F\,i!_F}{(n+j+r-1)!_F\,(n+2i-2)!_F\,j!_F\,(i-j)!_F}(-1)^{\frac{i(i-1)}{2}+ij+\frac{j(j+1)}{2}}.$$

3. Sketch of proof

We must simplify the following sum:

$$\sum_j L_{i,j}U_{j,k} = \sum_j \frac{(x;q)_{i+r}(q;q)_i(x;q)_{2j}}{(x;q)_{j+r}(x;q)_{i+j}(q;q)_j(q;q)_{i-j}}\alpha^{r(i-j)} \times \frac{x^jy^r}{(1-q)^r}\alpha^{r(j+k)+\frac{r(r-3)}{2}}q^{\frac{3(j-1)j}{2}}(-1)^j \times \frac{(x;q)_{k+r}(x;q)_{j-1}(q;q)_k(q;q)_r}{(x;q)_{j+k}(x;q)_{2j-1}(q;q)_{r-j}(q;q)_{k-j}}.$$

Apart from constant factors, we are left to compute

$$\sum_{j=0}^{\min\{i,k\}} x^j(-1)^jq^{\frac{3(j-1)j}{2}} \times \frac{(x;q)_{2j}(x;q)_{j-1}}{(x;q)_{j+r}(x;q)_{i+j}(x;q)_{j+k}(x;q)_{2j-1}(q;q)_j(q;q)_{i-j}(q;q)_{r-j}(q;q)_{k-j}}.$$

Zeilberger's algorithm [2] (the q-version of it) readily evaluates this as

$$\frac{(x;q)_{i+k+r}}{(x;q)_{i+r}(x;q)_{k+r}(x;q)_{i+k}(q;q)_r(q;q)_i(q;q)_k}.$$

Putting this together with the constant factors, this proves that $LU = M$.

4. The Lucas matrix

We briefly discuss the case of the matrix $\mathscr{M}$, where each F_i is replaced by the Lucas number L_i. We also need the notation $m!_L := L_1L_2\dots L_m$.

We write $L_m = \alpha^m+\beta^m = \alpha^m(1+q^m)$ and $L_{n+j} = y\alpha^j(1+xq^j)$, with $y=\alpha^n$ and $x=q^n$, when it comes to specializations.

Theorem 4. *The LU-decomposition $\mathscr{M} = \mathscr{L}\mathscr{U}$ is given by:*

$$\mathscr{U}_{i,j} = \frac{(-x;q)_{j+r}(-x;q)_{i-1}(q;q)_j(q;q)_r}{(q;q)_{j-i}(-x;q)_{i+j}(q;q)_{r-i}(-x;q)_{2i-1}} x^i y^r q^{\frac{3i(i-1)}{2}} \alpha^{r(i+j)+\frac{r(r-1)}{2}},$$

$$\mathscr{L}_{i,j} = \frac{(-x;q)_{i+r}(-x;q)_{2j}(q;q)_i}{(-x;q)_{j+r}(-x;q)_{i+j}(q;q)_j(q;q)_{i-j}} \alpha^{r(i-j)},$$

$$\mathscr{U}^{-1}_{i,j} = \frac{(-x;q)_{2j}(-x;q)_{i+j-1}(q;q)_{r-j}}{(-x;q)_{j-1}(-x;q)_{i+r}(q;q)_r(q;q)_i(q;q)_{j-i}}$$
$$\times x^{-j} y^{-r} q^{-j(j-1)-ij+\frac{i(i+1)}{2}} \alpha^{-r(i+j)+\frac{r(r-9)}{2}} (-1)^{i-j},$$

$$\mathscr{L}^{-1}_{i,j} = \frac{(-x;q)_{i+r}(-x;q)_{i+j-1}(q;q)_i}{(-x;q)_{j+r}(-x;q)_{2i-1}(q;q)_j(q;q)_{i-j}} q^{\frac{i(i-1)}{2}-ij+\frac{j(j+1)}{2}} \alpha^{r(i-j)}(-1)^{i-j}.$$

Theorem 5. *The specialized (Fibonacci/Lucas) forms are:*

$$\mathscr{U}_{i,j} = \frac{(n+j+r-1)!_L\,(n+i-2)!_L\,j!_F\,r!_F}{(n+i+j-1)!_L\,(n+2i-2)!_L\,(j-i)!_F\,(r-i)!_F} 5^i(-1)^{\frac{i(i-1)}{2}+ni},$$

$$\mathscr{L}_{i,j} = \frac{(n+i+r-1)!_L\,(n+2j-1)!_L\,i!_F}{(n+j+r-1)!_L\,(n+i+j-1)!_L\,j!_F\,(i-j)!_F},$$

$$\mathscr{U}^{-1}_{i,j} = \frac{(n+2j-1)!_L\,(n+i+j-2)!_L\,(r-j)!_F}{(n+j-2)!_L\,(n+i+r-1)!_L\,r!_F\,(j-i)!_F\,i!_F} 5^{-j}(-1)^{ij+\frac{i(i-1)}{2}+(n+1)j},$$

$$\mathscr{L}^{-1}_{i,j} = \frac{(n+i+r-1)!_L\,(n+i+j-2)!_L\,i!_F}{(n+j+r-1)!_L\,(n+2i-2)!_L\,j!_F\,(i-j)!_F}(-1)^{\frac{i(i+1)}{2}+ji+\frac{j(j-1)}{2}}.$$

Theorem 6. *The determinant of the matrix $\mathscr{M}$ is given by*

$$\det(\mathscr{M}) = \prod_{i=0}^{r} U_{i,i} = \prod_{i=0}^{r} \frac{(n+i+r-1)!_L\,(n+i-2)!_L\,i!_F\,r!_F}{(n+2i-1)!_L\,(n+2i-2)!_L\,(r-i)!_F} 5^i(-1)^{\frac{i(i-1)}{2}+ni}$$
$$= (r!_F)^{r+1} 5^{\binom{r+1}{2}} (-1)^{\binom{r+1}{3}+n\binom{r+1}{2}}. \quad \square$$

References

[1] L. Carlitz, Some Determinants Containing Powers of Fibonacci Numbers. *Fibonacci Quart.*, 4(2):129–134, 1966.

[2] M. Petkovšek, H. Wilf, and D. Zeilberger. *$A=B$*. A.K. Peters, Ltd., 1996.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH 7602, STELLENBOSCH, SOUTH AFRICA
E-mail address: hproding@sun.ac.za