THE RECIPROCAL SUPER CATALAN MATRIX

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ABSTRACT. The reciprocal super Catalan matrix has entries $\frac{i!(i+j)!j!}{(2i)!(2j)!}$. Explicit formulæ for its LU-decomposition, the LU-decomposition of its inverse, and some related matrices are obtained. For all results, q-analogues are also presented.

1. INTRODUCTION

For a given sequence $a_0, a_1, \ldots$, it is customary to study the associated Hankel matrix

$$
\begin{pmatrix}
a_0 & a_1 & a_2 & \ldots \\
a_1 & a_2 & a_3 & \ldots \\
a_2 & a_3 & a_4 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix},
$$

see [6]. There are also papers that concentrated on the reciprocals of a given sequence: the Hilbert matrix [2]

$$
\begin{pmatrix}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix},
$$

the Filbert matrix [10, 5]

$$
\begin{pmatrix}
\frac{1}{F_0} & \frac{1}{F_1} & \frac{1}{F_2} & \cdots \\
\frac{1}{F_1} & \frac{1}{F_2} & \frac{1}{F_3} & \cdots \\
\frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix},
$$

where the entries are Fibonacci numbers, and the reciprocal Pascal matrix [9]

$$
\begin{pmatrix}
\frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \cdots \\
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{1}{1} & \frac{1}{3} & \frac{1}{6} & \frac{1}{10} & \cdots \\
\frac{1}{1} & \frac{1}{4} & \frac{1}{10} & \frac{1}{20} & \cdots \\
\cdots & \cdots & \cdots & \cdots &
\end{pmatrix}.
$$

Strictly speaking, the latter instance is not the Hankel matrix of one sequence since the entries $\binom{i+j}{i}^{-1}$ depend on two parameters. All these matrices are either infinite or cut off after N rows and columns, depending on the context.

In this paper we concentrate on the matrix M of the reciprocal super Catalan numbers. The entries of M are $\left(\frac{(2i)!(2j)!}{i!j!(i+j)!}\right)^{-1}$, where we assume that the indices start at $(0,0)$, for convenience. These numbers (divided by 2) made their appearance into recent history via a paper by Gessel [3]. Setting $T(i,j) = \frac{(2i)!(2j)!}{2i!j!(i+j)!}$, it can be shown that these numbers are integers, and that $T(i,1)$ are the famous Catalan numbers. An interpretation of $T(i,2)$ in terms of blossom trees has been found by Schaeffer [11], and another in terms of cubic trees by Pippenger and Schleich [8]. An interpretation of $T(i,2)$ in terms of pairs of Dyck paths with restricted heights has been given by Gessel and Xin [4]. A more recent interpretation in terms of weighted paths is due to Allen and Gheorgiciuc [1]. Further papers can be found by various search engines.

Motivated by a recent paper by Richardson [9] who considered the reciprocal Pascal matrix with entries $\frac{i!j!}{(i+j)!}$ and factorizations of it, we moved here to the present instance.

We offer explicit expressions for the LU-decomposition $LU = M$, which leads to a formula for the determinant via $\prod_{0 \le i < n} U_{i,i}$. Further, we have expressions for L^{-1}, U^{-1}, for the LU-decomposition $AB = M^{-1}$, and for A^{-1}, B^{-1}. The latter expressions depend on the size N of the matrix M^{-1}. Via this decomposition, we find that the entries of M^{-1} are integers. This fact, together with an inherent and natural interest in the decompositions, was a main motivation for the current line of research.

As suggested by one referee, we also provide Cholesky decompositions:
We set $C \cdot \overline{C}^T = M$, $D \cdot \overline{D}^T = M^{-1}$, where the bar means complex conjugation, and the T means transposition.

We can also offer q-analogues of all these results. This means that in the definition of M, all factorials $n!$ are replaced by $(q)_n := (1-q)(1-q^2)\dots(1-q^n)$. The results are similar, with factorials accordingly replaced, but there are also powers of q present, which in the limit $q \to 1$ (the classical case) are hidden.

We also have analogous results for the matrix $\mathscr{M}$ with entries $\frac{(2i)!(2j)!}{i!j!(i+j)!}$. We use script letters $\mathscr{A}, \mathscr{B}, \mathscr{L}, \mathscr{U}, \mathscr{C}, \mathscr{D}$ for the respective quantities.

All these results were obtained by guessing using a computer. This is a time consuming process that requires some skills. Once the formulæ are obtained, proofs are certain combinatorial identities that are more or less routine, since there is software, largely due to D. Zeilberger [7], that can prove them. We will discuss a few of them here. The corresponding q-identities we only mention; they can also be proved in an automatic fashion. We would like to point out that these q-identities often have applications for identities about Fibonacci numbers. We cite the paper [5] to give the reader an idea.

It is likely that other combinatorial matrices allow a similar treatment; we plan to come back to this in the future.

2. Decomposition of M

We list here the formulæ that were found; all indices start at $(0,0)$.

$$L_{i,j} = \frac{i!i!i!(2j)!}{(2i)!(i-j)!j!j!j!},$$

$$L_{i,j}^{-1} = \frac{(-1)^{i+j}i!i!i!(2j)!}{(2i)!(i-j)!j!j!j!},$$

$$U_{i,j} = \frac{j!j!j!i!}{(2j)!(j-i)!(2i)!},$$

$$U_{i,j}^{-1} = \frac{(-1)^{i+j}(2j)!(2i)!}{j!(j-i)!i!i!i!},$$

$$A_{i,j} = \frac{(-1)^{i+j}(2i)!j!(N-j-1)!(2j+1)}{i!(N-i-1)!(i+j+1)!(i-j)!},$$

$$A_{i,j}^{-1} = \frac{(N-j-1)!(i+j)!j!}{(N-i-1)!i!(i-j)!(2j)!},$$

$$B_{i,j} = \frac{(-1)^{i+j}(2j)!(N+i)!}{j!(N-j-1)!i!(i+j+1)!(j-i)!},$$

$$B_{i,j}^{-1} = \frac{(2j+1)j!(N-i-1)!i!(i+j)!}{(N+j)!(2i)!(j-i)!},$$

$$C_{i,j} = \frac{i!i!i!}{(2i)!(i-j)!j!},$$

$$C_{i,j}^{-1} = \frac{i!(2j)!}{(i-j)!j!j!j!},$$

$$D_{i,j} = \frac{(2i)!}{(N-1-i)!(i-j)!i!}\sqrt{(N+j)!(N-1-j)!(2j+1)},$$

$$D_{i,j}^{-1} = \frac{(i+j)!(N-1-j)!j!}{(i-j)!(2j)!}\sqrt{\frac{(2i+1)}{(N+i)!(N-1-i)!}}.$$

This leads as a corollary to a formula for the determinant of M_N (we need the finite version here):

$$\det M_N = \prod_{0 \le i < N} \frac{i!^4}{(2i)!^2} = \prod_{0 \le i < N} \binom{2i}{i}^{-2}.$$

Some of the identities that needs to be proved are written out in some detail. Note that instead of proving that $AB = M^{-1}$ it is more convenient to prove the equivalent $B^{-1}A^{-1} = M$.

We use the handy *Iversion notation*: $[\![P]\!]$ is 1 if P is true, and 0 otherwise.

$$\sum_k L_{i,k}L_{k,j}^{-1} = \frac{i!i!i!(2j)!}{(2i)!j!j!j!}\sum_k \frac{(-1)^{k+j}}{(i-k)!(k-j)!}$$

$$= \frac{i!i!i!(2j)!}{(2i)!j!j!j!(i-j)!}\sum_k (-1)^{k-j}\binom{i-j}{k-j}$$

$$= \frac{i!i!i!(2j)!}{(2i)!j!j!j!(i-j)!}[\![i=j]\!] = [\![i=j]\!].$$

$$\sum_k U_{i,k}U_{k,j}^{-1} = \sum_k \frac{i!}{(k-i)!(2i)!}\frac{(-1)^{k+j}(2j)!}{j!(j-k)!}$$

$$= \frac{i!(2j)!}{j!(2i)!}\sum_k \frac{(-1)^{k+j}}{(k-i)!(j-k)!}$$

$$= \frac{i!(2j)!}{j!(2i)!(j-i)!}[\![i=j]\!] = [\![i=j]\!].$$

$$\sum_k A_{i,k}A_{k,j}^{-1} = \frac{(2i)!(N-j-1)!j!}{i!(N-i-1)!(2j)!}\sum_k \frac{(-1)^{i+k}(2k+1)(k+j)!}{(i+k+1)!(i-k)!(k-j)!} = [\![i=j]\!].$$

$$\sum_k B_{i,k}B_{k,j}^{-1} = \frac{(N+i)!(2j+1)j!}{i!(N+j)!}\sum_k \frac{(-1)^{i+k}(k+j)!}{(i+k+1)!(k-i)!(j-k)!}$$

$$= \frac{(N+i)!(2j+1)j!}{i!(N+j)!}[\![i=j]\!]\frac{1}{2i+1} = [\![i=j]\!].$$

$$\sum_k B_{i,k}^{-1}A_{k,j}^{-1} = \frac{(N-i-1)!i!(N-j-1)!j!}{(2i)!(i-j)!(2j)!}\sum_k \frac{(2k+1)(i+k)!(k+j)!}{(N+k)!(k-i)!(N-k-1)!}$$

$$= \frac{(N-i-1)!i!(N-j-1)!j!}{(2i)!(2j)!}\frac{(i+j)!}{(N-i-1)!(N-j-1)!}$$

$$= \frac{i!j!(i+j)!}{(2i)!(2j)!}.$$

Now we compute an arbitrary entry of AB:

$$\sum_k A_{i,k}B_{k,j} = \frac{(-1)^{i+j}(2i)!(2j)!}{i!(N-i-1)!j!(N-j-1)!}\sum_k \frac{(N-k-1)!(N+k)!}{(i+k+1)!(i-k)!(k+j+1)!(j-k)!}$$

$$\times [(i+k+1)-(i-k)]$$

$$= \frac{(-1)^{i+j}(2i)!(2j)!}{i!(N-i-1)!j!(N-j-1)!}\sum_k \frac{(N-k-1)!(N+k)!}{(i+k)!(i-k)!(k+j+1)!(j-k)!}$$

$$- \frac{(-1)^{i+j}(2i)!(2j)!}{i!(N-i-1)!j!(N-j-1)!}\sum_k \frac{(N-k-1)!(N+k)!}{(i+k+1)!(i-k-1)!(k+j+1)!(j-k)!}$$

$$= \frac{(-1)^{i+j}(2i)!(2j)!}{i!j!(i+j)!}\sum_k \binom{N-k-1}{i-k}\binom{N+k}{j+k+1}\binom{i+j}{i+k}$$

$$- \frac{(-1)^{i+j}(2i)!(2j)!}{i!j!(i+j)!}\sum_k \binom{N-k-1}{j-k}\binom{N+k}{i+k+1}\binom{i+j}{j+k+1}.$$

This shows that the entries of M^{-1} are integers, since the factors in front are super Catalan numbers, which are integers. A similar question about the reciprocal Pascal matrix was answered in the affirmative by Richardson [9].

3. q-ANALOGUES OF THE DECOMPOSITION OF M

We assume that M has entries $\left(\frac{(q)_{2i}(q)_{2j}}{(q)_i(q)_{i+j}(q)_j}\right)^{-1}$.

$$L_{i,j} = \frac{(q)_i(q)_i(q)_i(q)_{2j}}{(q)_{2i}(q)_{i-j}(q)_j(q)_j(q)_j},$$

$$L_{i,j}^{-1} = \frac{(-1)^{i+j}q^{(i-j)(i-j-1)/2}(q)_i(q)_i(q)_i(q)_{2j}}{(q)_{2i}(q)_{i-j}(q)_j(q)_j(q)_j},$$

$$U_{i,j} = \frac{q^{i^2}(q)_i(q)_j(q)_j(q)_j}{(q)_{2i}(q)_{j-i}(q)_{2j}},$$

$$U_{i,j}^{-1} = \frac{(-1)^{i+j}q^{i(i+1)/2-ij-j(j+1)/2}(q)_{2i}(q)_{2j}}{(q)_i(q)_i(q)_i(q)_{j-i}(q)_j},$$

$$A_{i,j} = \frac{(-1)^{i+j}q^{(i-j)(i+j+3)/2-N(i-j)}(q)_{2i}(q)_j(q)_{N-j-1}(1-q^{2j+1})}{(q)_i(q)_{N-i-1}(q)_{i+j+1}(q)_{i-j}},$$

$$A_{i,j}^{-1} = \frac{q^{(i-j)(i-N+1)}(q)_{N-j-1}(q)_{i+j}(q)_j}{(q)_i(q)_{N-i-1}(q)_{i-j}(q)_{2j}},$$

$$B_{i,j} = \frac{(-1)^{i+j}q^{(j+1)(j+2)/2-N(i+j+1)+3i(i+1)/2}(q)_{2j}(q)_{N+i}}{(q)_i(q)_{N-j-1}(q)_j(q)_{i+j+1}(q)_{j-i}},$$

$$B_{i,j}^{-1} = \frac{q^{(i+j+1)(N-j-1)}(q)_i(1-q^{2j+1})(q)_j(q)_{N-i-1}(q)_{i+j}}{(q)_{N+j}(q)_{2i}(q)_{j-i}},$$

$$C_{i,j} = \frac{q^{j^2/2}(q)_i(q)_i(q)_i}{(q)_{2i}(q)_{i-j}(q)_j},$$

$$C_{i,j}^{-1} = \frac{q^{-i/2-ij+j(j+1)/2}(q)_i(q)_{2j}}{(q)_{i-j}(q)_j(q)_j(q)_j},$$

$$D_{i,j} = \frac{q^{i(i+3)/2-Ni-N/2+j^2/2+1/2}(q)_{2i}}{(q)_{N-1-i}(q)_{i-j}(q)_i}\sqrt{(q)_{N+j}(q)_{N-1-j}(1-q^{2j+1})},$$

$$D_{i,j}^{-1} = \frac{q^{-i/2-ij-j+Nj+N/2-1/2}(q)_{i+j}(q)_{N-1-j}(q)_j}{(q)_{i-j}(q)_{2j}}\sqrt{\frac{(1-q^{2i+1})}{(q)_{N+i}(q)_{N-1-i}}}.$$

4. DECOMPOSITION OF $\mathscr{M}$

Recall that $\mathscr{M}$ has entries $\frac{(2i)!(2j)!}{i!j!(i+j)!}$.

$$\mathscr{L}_{i,j} = \frac{(2i)!}{(i+j)!(i-j)!},$$

$$\mathscr{L}_{i,j}^{-1} = \frac{2(-1)^{i+j}i(i+j-1)!}{(i-j)!(2j)!} \quad \text{for } j \geq 1, \quad \mathscr{L}_{i,0}^{-1} = 2(-1)^i \quad \text{for } i \geq 1, \quad \mathscr{L}_{0,0}^{-1} = 1,$$

$$\mathscr{U}_{i,j} = \frac{2(-1)^i(2j)!}{(i+j)!(j-i)!} \quad \text{for } i \geq 1, \quad \mathscr{U}_{0,j} = \frac{(2j)!}{j!j!},$$

$$\mathscr{U}_{i,j}^{-1} = \frac{(-1)^i j(i+j-1)!}{(j-i)!(2i)!} \quad \text{for } i \geq 1, \quad \mathscr{U}_{0,j}^{-1} = 1,$$

$$\mathscr{A}_{i,j} = \frac{(-1)^{i+j}(N-j-1)!(2j)!(N+i-1)!i!}{(N+j-1)!j!(N-i-1)!(2i)!(i-j)!},$$

$$\mathscr{A}_{i,j}^{-1} = \frac{(N-j-1)!(2j)!(N+i-1)!i!}{(N+j-1)!j!(N-i-1)!(2i)!(i-j)!},$$

$$\mathscr{B}_{i,j} = \frac{(-1)^{N+j+1}(N+j-1)!i!j!}{(j-i)!(N-j-1)!(2j)!(2i)!},$$

$$\mathscr{B}_{i,j}^{-1} = \frac{(-1)^{N+j+1}(2j)!(N-i-1)!(2i)!}{(N+i-1)!i!j!(j-i)!},$$

$$\mathscr{C}_{i,j} = \frac{\mathbf{i}^j\sqrt{2}(2i)!}{(i+j)!(i-j)!} \quad \text{for } j \geq 1, \quad \mathscr{C}_{i,0} = \frac{(2i)!}{i!},$$

$$\mathscr{C}_{i,j}^{-1} = \frac{\mathbf{i}^i\sqrt{2}(i+j-1)!i}{(2j)!(i-j)!} \quad \text{for } i \geq 1, \quad \mathscr{C}_{0,0}^{-1} = 1,$$

$$\mathscr{D}_{i,j} = \frac{\mathbf{i}^{N-j-1}(N+i-1)!i!}{(N-i-1)!(i-j)!(2i)!}\sqrt{\frac{(N-j-1)!}{(N+j-1)!}},$$

$$\mathscr{D}_{i,j}^{-1} = \frac{\mathbf{i}^{N-i-1}(N-j-1)!(2j)!}{(i-j)!(N+j-1)!j!}\sqrt{\frac{(N+i-1)!}{(N-i-1)!}}.$$

Here are a few computations validating the claims:

$$\sum_k \mathscr{A}_{i,k}\mathscr{A}_{k,j}^{-1} = \frac{(N+i-1)!i!(N-j-1)!(2j)!}{(N-i-1)!(2i)!(N+j-1)!j!(i-j)!}\sum_k(-1)^{i-k}\binom{i-j}{i-k}$$

$$= \frac{(N+i-1)!i!(N-j-1)!(2j)!}{(N-i-1)!(2i)!(N+j-1)!j!(i-j)!}[\![i=j]\!] = [\![i=j]\!],$$

$$\sum_k \mathscr{B}_{i,k}\mathscr{B}_{k,j}^{-1} = \frac{i!(2j)!}{(2i)!j!(j-i)!}\sum_k(-1)^{j-k}\binom{j-i}{j-k} = [\![i=j]\!].$$

5. q-ANALOGUES OF THE DECOMPOSITION OF $\mathscr{M}$

We assume that $\mathscr{M}$ has entries $\frac{(q)_{2i}(q)_{2j}}{(q)_i(q)_{i+j}(q)_j}$.

$$\mathscr{L}_{i,j} = \frac{(q)_{2i}}{(q)_{i+j}(q)_{i-j}},$$

$$\mathscr{L}_{i,j}^{-1} = \frac{q^{(i-j)(i-1-j)/2}(-1)^{i+j}(1-q^{2i})(q)_{i+j-1}}{(q)_{i-j}(q)_{2j}} \quad \text{for } j \geq 1,$$

$$\mathscr{L}_{i,0}^{-1} = (1+q^i)(-1)^i q^{i(i-1)/2} \quad \text{for } i \geq 1, \quad \mathscr{L}_{0,0} = 1,$$

$$\mathscr{U}_{i,j} = \frac{(-1)^i q^{i(3i-1)/2}(1+q^i)(q)_{2j}}{(q)_{i+j}(q)_{j-i}} \quad \text{for } i \geq 1, \quad \mathscr{U}_{0,j} = \frac{(q)_{2j}}{(q)_j(q)_j},$$

$$\mathscr{U}_{i,j}^{-1} = \frac{(-1)^i q^{i(i+1)/2-ij-j^2}(1-q^j)(q)_{i+j-1}}{(q)_{j-i}(q)_{2i}} \quad \text{for } i \geq 1, \quad \mathscr{U}_{0,j}^{-1} = q^{-j^2},$$

$$\mathscr{A}_{i,j} = \frac{(-1)^{i+j} q^{(i+1)(i+2)/2-(j+2)(j+1)/2+(j-i)N}(q)_{N-j-1}(q)_{2j}(q)_{N+i-1}(q)_i}{(q)_{N+j-1}(q)_j(q)_{N-i-1}(q)_{2i}(q)_{i-j}},$$

$$\mathscr{A}_{i,j}^{-1} = \frac{q^{(i-j)(i-N+1)}(q)_{N-j-1}(q)_{2j}(q)_{N+i-1}(q)_i}{(q)_{N+j-1}(q)_j(q)_{N-i-1}(q)_{2i}(q)_{i-j}},$$

$$\mathscr{B}_{i,j} = \frac{q^{(j+1)(j+2)/2-N(N-1)/2-jN+i^2-1}(q)_{N+j-1}(q)_i(q)_j}{(q)_{j-i}(q)_{N-j-1}(q)_{2j}(q)_{2i}},$$

$$\mathscr{B}_{i,j}^{-1} = \frac{(-1)^{N+j+1} q^{i-ij-j(j+1)/2+iN+N(N-1)/2}(q)_{2j}(q)_{N-i-1}(q)_{2i}}{(q)_{N+i-1}(q)_i(q)_j(q)_{j-i}},$$

$$\mathscr{C}_{i,j} = \frac{\mathbf{i}^j\sqrt{1+q^j}\, q^{j(3j-1)/4}(q)_{2i}}{(q)_{i+j}(q)_{i-j}} \quad \text{for } j \geq 1, \quad \mathscr{C}_{i,0} = \frac{(q)_{2i}}{(q)_i},$$

$$\mathscr{C}_{i,j}^{-1} = \frac{\mathbf{i}^i\sqrt{1+q^i}\, q^{-i(i+1)/4-ij+j(j+1)/2}(q)_{i+j-1}(1-q^i)}{(q)_{2j}(q)_{i-j}} \quad \text{for } i \geq 1, \quad \mathscr{C}_{0,0}^{-1} = 1,$$

$$\mathscr{D}_{i,j} = \frac{\mathbf{i}^{N-j-1} q^{-N(N-1)/4+i(i+3)/4+N(j-2i)/2+j(j-3)/4}(q)_{N+i-1}(q)_i}{(q)_{N-i-1}(q)_{i-j}(q)_{2i}}\sqrt{\frac{(q)_{N-j-1}}{(q)_{N+j-1}}},$$

$$\mathscr{D}_{i,j}^{-1} = \frac{\mathbf{i}^{N-i-1} q^{i(i+1)/4+N(2j-i)/2-ij+N(N-1)/4}(q)_{N-j-1}(q)_{2j}}{(q)_{i-j}(q)_{N+j-1}(q)_j}\sqrt{\frac{(q)_{N+i-1}}{(q)_{N-i-1}}}.$$

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