PARTIAL SKEW MOTZKIN PATHS

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Abstract. Motzkin paths consist of up-steps, down-steps, level-steps, and never go below the x-axis. They return to the x-axis at the end. The concept of skew Dyck path [1] is transferred to skew Motzkin paths, namely, a left step $(-1,-1)$ is additionally allowed, but the path is not allowed to intersect itself. The enumeration of these combinatorial objects was known [6]; here, using the kernel method, we extend the results by allowing them to end at a prescribed level j. The approach is completely based on generating functions.

Asymptotics of the total number of objects as well as the average height are also given.

1. Introduction

A Dyck path has up-steps and down-steps of one unit each, and cannot go into negative territory. Usually, one considers returns to the x-axis at the end, but for partial paths, this is not required. A Motzkin path allows additional flat (horizontal) steps of unit length. A skew path allows 'left' step $(-1,-1)$ as well, but the path is not allowed to intersect itself. We prefer 'red' steps $(1,-1)$, see our analysis in [5]. For Motzkin paths, some analysis as provided in [6]. Here, we provide further analysis that allows to consider partial paths as well, so we don't need to land at the x-axis. It uses the kernel method [4]. Apart from being not below the x-axis, the restrictions are that a left (red) step cannot follow or preceed an up-step.

The situation is best described by a graph (state-diagram); see Figure 1.

In further sections, the asymptotic equivalent for the number of skew Motzkin paths of given size is derived, as well as the *height*, meaning that the generating function of paths with a bounded height (bounded by H) is given, as well as the average height, which is approximately $\mathsf{const}\cdot\sqrt{n}$, which is typical for families of paths.

2. Generating functions for skew Motzkin paths

We translate the state diagram accordingly; f_j, g_j, h_j, k_j are generating functions in the variable z (marking the length of the path), ending at level j. The four families are related to the four layers of states.

$$\begin{aligned} f_{j+1} &= zf_j + zg_j + zh_j, \ \ f_0 = 1, \\ g_j &= zf_{j+1} + zg_{j+1} + zh_{j+1} + zk_{j+1}, \\ h_j &= zf_j + zg_j + zh_j + zk_j, \\ k_j &= zg_{j+1} + zh_{j+1} + zk_{j+1}. \end{aligned}$$

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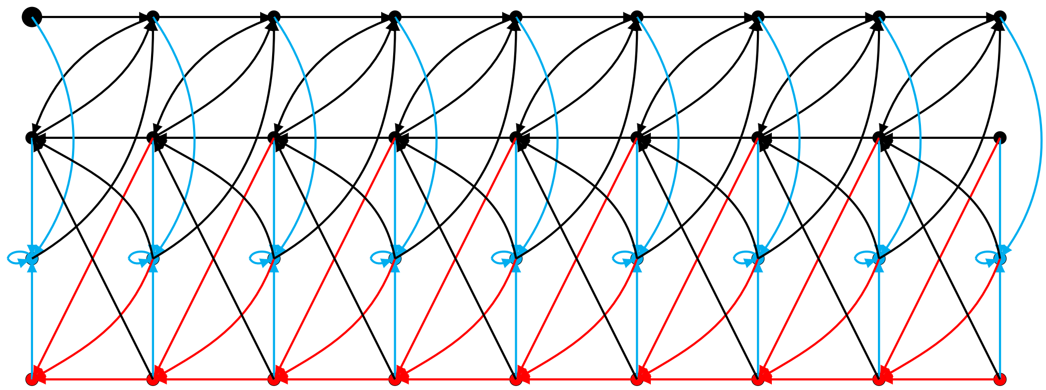

FIGURE 1. Four layers of states according to the type of steps leading to them. Traditional up-steps and down-steps are black, level-steps are blue, and left steps are red.

Now we introduce bivariate generating functions, namely

$$F(u) := \sum_{j\ge0} f_j u^j,\ G(u) := \sum_{j\ge0} g_j u^j,\ H(u) := \sum_{j\ge0} h_j u^j,\ K(u) := \sum_{j\ge0} k_j u^j.$$

The recursions then take this form:

$$\begin{aligned}
F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\
uG(u) &= zF(u) + zG(u) + zH(u) + zK(u) - z - zg_0 - zh_0 - zk_0,\\
H(u) &= zF(u) + zG(u) + zH(u) + zK(u),\\
uK(u) &= zG(u) + zH(u) + K(u) - zg_0 + zh_0 + zk_0.
\end{aligned}$$

Solving the system,

$$\begin{aligned}
F(u) &= \frac{\mathscr{F}}{2z - u + zu - z^2u + zu^2 - z^3 - z^3u},\\
G(u) &= \frac{\mathscr{G}}{2z - u + zu - z^2u + zu^2 - z^3 - z^3u},\\
H(u) &= \frac{\mathscr{H}}{2z - u + zu - z^2u + zu^2 - z^3 - z^3u},\\
K(u) &= \frac{\mathscr{K}}{2z - u + zu - z^2u + zu^2 - z^3 - z^3u},
\end{aligned}$$

with $\mathscr{F} = -z^3 + 2z - u + zu + z^2u + z^2ug_0 + z^2uh_0 + z^2uk_0 + z^3ug_0 + z^3uh_0 + z^3uk_0$, $\mathscr{G} = -z^2h_0 + z^4 + z^4k_0 - z^2ug_0 - z^2u - z^2k_0 - z^2 - z^2uh_0 - z^2uk_0 + z^4h_0 + z^3 - z^2g_0 + zh_0 + zg_0 + zk_0 + z^4g_0$, $\mathscr{H} = -z^4 + 2z^2g_0 + 2z^2h_0 + 2z^2k_0 + 2z^2 - zu - z^4g_0 - z^4h_0 - z^4k_0 - z^3ug_0 - z^3uh_0 - z^3uk_0$, $\mathscr{K} = zg_0 - z^3 - z^2g_0 - z^2h_0 + zk_0 - z^2k_0 - z^3g_0 - z^3h_0 - z^3k_0$.

One cannot immediately insert $u = 0$ to identify the constants, but one can use the kernel method. For that, one factorises the denominator:

$$2z - u + zu - z^2u + zu^2 - z^3 - z^3u = z(u - u_1)(u - u_2)$$

with

$$u_1 = \frac{1 - z + z^2 + z^3 + (1+z)W}{2z}, \quad u_2 = \frac{1 - z + z^2 + z^3 - (1+z)W}{2z}$$

and

$$W = \sqrt{(1-z)(1-3z-z^2-z^3)} = \sqrt{1-4z+2z^2+z^4}.$$

Since $u_2 \sim 2z$ for small z, $u - u_2$ is a 'bad' factor and must cancel from both, numerator and denominator. This yields

$$\begin{aligned}
F(u) &= \frac{-1 + z + z^2 + z^2 g_0 + z^2 h_0 + z^2 k_0 + z^3 g_0 + z^3 h_0 + z^3 k_0}{z(u - u_1)},\\
G(u) &= \frac{-z^2 - z^2 g_0 - z^2 h_0 - z^2 k_0}{z(u - u_1)},\\
H(u) &= \frac{-z - z^3 g_0 - z^3 h_0 - z^3 k_0}{z(u - u_1)},\\
K(u) &= \frac{-z^2 g_0 - z^2 h_0 - z^2 k_0}{z(u - u_1)}.
\end{aligned}$$

Now we can plug in $u = 0$ and identify the constants:

$$\begin{aligned}
g_0 &= \frac{-z^5 + z^3 W - z^2 - zW + 3z - 1 + W}{2z^2\left(-2 + z^2\right)},\\
h_0 &= -\frac{-z^2 + 2z - 1 + W}{2z},\\
k_0 &= -\frac{-z^4 + z^3 + z^2 W + zW - 3z + 1 - W}{2z^2\left(-2 + z^2\right)}.
\end{aligned}$$

Adding these quantities yields

$$1 + g_0 + h_0 + k_0 = -\frac{-z^2 + 2z - 1 + W}{2z^2},$$

which is the generating function of the number of skew Motzkin paths (returning to the x-axis); the series expansion is

$$1 + z + 2z^2 + 5z^3 + 13z^4 + 35z^5 + 97z^6 + 275z^7 + 794z^8 + 2327z^9 + 6905z^{10} + 20705z^{11} + \cdots,$$

as already given in [6], the coefficients are sequence A82582 in [7].

We further get

$$\begin{aligned}
F(u) &= \frac{-1 + z - z^2 - z^3 + u_2 z}{z(u - u_1)},\\
G(u) &= \frac{(z - u_2)}{(u - u_1)(1+z)},\\
H(u) &= \frac{-1 - z + 2z^2 + z^3 - z u_2}{(u - u_1)(1+z)},
\end{aligned}$$

$$K(u) = \frac{z^2 + 2z - u_2}{(u - u_1)(1 + z)}.$$

Altogether,

$$F(z) + G(z) + H(z) + K(z) = \frac{-1 - z + 2z^2 + z^3 - zu_2}{z(u - u_1)(1 + z)}$$

and

$$[u^j]\big(F(z) + G(z) + H(z) + K(z)\big) = \frac{1 + z - 2z^2 - z^3 + zu_2}{z(1 + z)u_1^{j+1}},$$

which is the generating function of partial skew Motzkin paths, landing on level j.

Here are the examples for $j = 1, 2, 3, 4$ (leading terms only):

$$z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + 102z^6 + 295z^7 + 866z^8 + 2574z^9 + 7730z^{10} + 23419z^{11},$$
$$z^2 + 3z^3 + 9z^4 + 26z^5 + 77z^6 + 230z^7 + 694z^8 + 2110z^9 + 6459z^{10} + 19890z^{11} + 61577z^{12},$$
$$z^3 + 4z^4 + 14z^5 + 45z^6 + 143z^7 + 451z^8 + 1421z^9 + 4478z^{10} + 14129z^{11} + 44654z^{12},$$
$$z^4 + 5z^5 + 20z^6 + 71z^7 + 242z^8 + 806z^9 + 2653z^{10} + 8670z^{11} + 28213z^{12}.$$

One can also substitute $u = 1$, which means that *all* partial skew Motzkin paths are counted with respect to length, regardless on which level they end:

$$\frac{2 - 3z - 7z^2 - z^3 + z^4 - (2 + z)\,(1 + z)\,W}{2z\,(1 + z)\,(2z^2 + 3z - 1)}.$$

The series expansion is

$$1 + 2z + 5z^2 + 14z^3 + 40z^4 + 117z^5 + 348z^6 + 1049z^7 + 3196z^8 + 9823z^9 + 30413z^{10} + \cdots$$

3. Counting flat and left (red) steps

Using two extra variables t and w, we can count the number of flat resp. left steps in a skew Motzkin path. The recursions are self-explanatory.

$$\begin{aligned}
f_{j+1} &= zf_j + zg_j + zh_j,\;\; f_0 = 1,\\
g_j &= zf_{j+1} + zg_{j+1} + zh_{j+1} + zk_{j+1},\\
h_j &= ztf_j + ztg_j + zth_j + ztk_j,\\
k_j &= zwg_{j+1} + zwh_{j+1} + zwk_{j+1}.
\end{aligned}$$

Again, here is the system for the multi-variate generating functions;

$$\begin{aligned}
F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\
uG(u) &= zF(u) + zG(u) + zH(u) + zK(u) - z - zg_0 - zh_0 - zk_0,\\
H(u) &= ztF(u) + ztG(u) + ztH(u) + ztK(u),\\
uK(u) &= zwG(u) + zwH(u) + zwK(u) - zwg_0 + zwh_0 + zwk_0.
\end{aligned}$$

And following a similar procedure as before we get

$$\begin{aligned}1+g_0+h_0+k_0&=\frac{-zw+u_2}{z(1+wt)}\\&=1+tz+(t^2+1)z^2+(tw+3t+t^3)z^3+(2+6t^2+w+3wt^2+t^4)z^4+\cdots;\end{aligned}$$

the quantity u_2 is now

$$u_2=\frac{1-tz+wz^2+twz^3-\sqrt{(1-z^2w)(1-2\,tz+(t^2-4-w)\,z^2-2twz^3-wt^2z^4)}}{2z}.$$

Quantities like $F(u)$, $G(u)$, $H(u)$, $K(u)$ can also be computed easily, following the approach from the previous section.

4. Asymptotics for the number of skew Motzkin paths

We must analyze the generating function

$$\mathscr{SM}=\frac{(1-z)^2-\sqrt{(1-z)(1-3z-z^2-z^3)}}{2z^2}$$

which is of the sqrt-type [2] around the singularity ρ closest to the origin, which we call ρ. It is a solution of $1-3z-z^2-z^3=0$ and an algebraic number of order 3. We confine ourselves to real numbers to keep the expressions shorter. We expand everything around $\rho=0.29559774252208477098099 6$. We find

$$\begin{aligned}\mathscr{SM}&\sim\frac{(1-\rho)^2-\sqrt{2.7142940417132890604(\rho-z)}}{2\rho^2}\\&\sim 2.8392867552141611323-9.4274931376001571585\sqrt{\rho-z}\\&\sim 2.8392867552141611323-5.1256244361431546460\sqrt{1-z/\rho}.\end{aligned}$$

Singularity analysis of generating function [2] gives the estimate

$$[z^n]\mathscr{SM}\sim 5.1256244361431546460\frac{1}{2\sqrt{\pi}}\rho^{-n}n^{-3/2}.$$

The error at $n=100$ is about 3%. This is to be expected by this type of approximation.

5. Skew Motzkin paths of bounded height

Now we introduce a parameter H and don't allow the path to reach any level higher than H. We can still work with the system

$$\begin{aligned}f_{j+1}&=zf_j+zg_j+zh_j,\ 0\le j\le H-1,\ f_0=1,\\g_j&=zf_{j+1}+zg_{j+1}+zh_{j+1}+zk_{j+1},\ 0\le j<H,\\h_j&=zf_j+zg_j+zh_j+zk_j,\ 0\le j\le H,\\k_j&=zg_{j+1}+zh_{j+1}+zk_{j+1},\ 0\le j<H.\end{aligned}$$

This is now a finite linear system, and we are only interested in paths that return to the x-axis. For a given H, we write $s[H]=f_0+g_0+h_0+k_0$ and let Maple compute these quantities for the first 20 values of H.

Both, numerator and denominator of $s[H]$ satisfy the recursion

$$X_{n+2}+(-1+z-z^2-z^3)X_{n+1}+(2z^2-z^4)X_n=0.$$

Thus, adjusting this to the initial conditions, we get

$$s[n]=\frac{A_o(1+z^3+z^2-z+\omega)^n+B_o(1+z^3+z^2-z-\omega)^n}{A_u(1+z^3+z^2-z+\omega)^n+B_u(1+z^3+z^2-z-\omega)^n}$$

with

$$\begin{aligned}
A_o&=(z^3+z^2+3z-1)(z+1)+(z-1)\omega,\\
B_o&=(z^3+z^2+3z-1)(z+1)-(z-1)\omega,\\
A_u&=(1-z^2)(z^3+z^2+3z-1)+\frac{z^3-z^2+3z-1}{1-z}\omega,\\
B_u&=(1-z^2)(z^3+z^2+3z-1)-\frac{z^3-z^2+3z-1}{1-z}\omega,\\
\omega&=\sqrt{z^6+2z^5+3z^4-5z^2-2z+1}=(1+z)W.
\end{aligned}$$

When n goes to infinity, the second terms go away, and we are left with

$$s[\infty]=\frac{A_o}{A_u}=\frac{(1-z)^2-\sqrt{(1-z)(1-3z-z^2-z^3)}}{2z^2}=\mathscr{S}\mathscr{M},$$

as expected. Now we consider $s[>n]$, the generating function of skew Motzkin paths of height $>n$. Taking differences, we find

$$\begin{aligned}
s[>n]=s[\infty]-s[n]&=\frac{A_oB_u-A_uB_o}{A_u}\frac{(1+z^3+z^2-z-\omega)^n}{A_u(1+z^3+z^2-z+\omega)^n+B_u(1+z^3+z^2-z-\omega)^n}\\
&\sim\frac{A_oB_u-A_uB_o}{A_u^2}\frac{\left(\dfrac{1+z^3+z^2-z-\omega}{1+z^3+z^2-z+\omega}\right)^n}{1-\left(\dfrac{1+z^3+z^2-z-\omega}{1+z^3+z^2-z+\omega}\right)^n}.
\end{aligned}$$

A computer computation leads to (always in the neighbourhood of $z=\rho$)

$$\frac{A_oB_u-A_uB_o}{A_u^2}\sim 18.854986275200314363\sqrt{\rho-z}.$$

Now we approximate:

$$\begin{aligned}
\frac{1+z^3+z^2-z-\omega}{1+z^3+z^2-z+\omega}&\sim\frac{0.81760902991166091601-2.1345121404980002137\sqrt{\rho-z}}{0.81760902991166091601+2.1345121404980002137\sqrt{\rho-z}}\\
&\sim\frac{1-2.6106758394395728799\sqrt{\rho-z}}{1+2.6106758394395728799\sqrt{\rho-z}}\\
&\sim 1-5.2213516788791457598\sqrt{\rho-z}\\
&\sim\exp\bigl(-5.2213516788791457598\sqrt{\rho-z}\bigr)=e^{-t},
\end{aligned}$$

for convenience. For the average height, we need apart from the leading factor,

$$\sum_{h\ge0} s[>h] \sim \sum_{h\ge0} \frac{e^{-th}}{1-e^{-th}}.$$

Since we only compute the leading term of the asymptotics of the average height, we might start the sum at $h \ge 1$, and expand the geometric series:

$$\sum_{h\ge1} s[>h] \sim \sum_{h,k\ge1} e^{-thk} = \sum_{k\ge1} d(k)e^{-kt} \sim -\frac{\log t}{t},$$

with $d(k)$ being the number of divisors of k. This type of analysis, although having been done often before, has been described in much detail in [3]. Together with the factor in front, we are at

$$\begin{aligned}
&-18.854986275200314363\sqrt{\rho-z}\frac{\log\sqrt{\rho-z}}{5.2213516788791457598\sqrt{\rho-z}}\\
&= -18.854986275200314363\frac{\log\sqrt{\rho-z}}{5.2213516788791457598}\\
&= -1.8055656307800996608\log(\rho-z)\\
&\sim -1.8055656307800996608\log(1-z/\rho).
\end{aligned}$$

Singularity analysis [2] gives the following estimate for the coefficient of z^n:

$$1.8055656307800996608\frac{\rho^{-n}}{n}.$$

For the average height we need to normalize, which is to divide by the total number of skew Motzkin numbers of size n:

$$\frac{1.8055656307800996608\frac{\rho^{-n}}{n}}{5.1256244361431546460\frac{1}{2\sqrt{\pi}}\rho^{-n}n^{-3/2}} = 0.70452513767814089508\sqrt{\pi n}.$$

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