CONTINUED FRACTION EXPANSIONS FOR \( q \)-TANGENT AND \( q \)-COTANGENT FUNCTIONS

HELMUT PRODINGER

Abstract. For 3 different versions of \( q \)-tangent resp. \( q \)-cotangent functions, we compute the continued fraction expansion explicitly, by guessing the relative quantities and proving the recursive relation afterwards. It is likely that these are the only instances with a “nice” expansion.

1. Introduction

In this paper, we consider the functions

\[
F(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n + 1]_q!} q^{dn^2},
\]

\[
G(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n]_q!} q^{dn^2}.
\]

We use standard \( q \)-notation:

\[
[n]_q := \frac{1 - q^n}{1 - q}, \quad [n]_q! := [1]_q[2]_q \ldots [n]_q.
\]

For \( d = 0, 1, 2 \), we will find the following continued fraction expansions:

\[
\frac{zF(z)}{G(z)} = \frac{z}{a_1 + \frac{z}{a_2 + \frac{z}{a_3 + \frac{z}{\ddots}}}}
\]

(Replacing \( z \) by \( z^2 \), we get \( z \) times a \( q \)-tangent function.)

\[
\frac{zG(z)}{F(z)} = \frac{z}{a_1 + \frac{z}{a_2 + \frac{z}{a_3 + \frac{z}{\ddots}}}}
\]

(Replacing \( z \) by \( z^2 \), we get \( z^3 \) times a \( q \)-cotangent function.)

These \( q \)-trigonometric functions are variants of Jackson’s, see [2].

The instance \( d = 0 \) of the \( q \)-tangent appeared in [3], and the instance \( d = 1 \) in [1] and [4]. The instance \( d = 2 \) as well as the \( q \)-cotangent expansions seem to be new.

Date: June 12, 2008.

Key words and phrases. \( q \)-tangent, \( q \)-cotangent, continued fraction.
Computer experiments indicate that, apart from trivial variations, these are the only cases where we get “nice” coefficients $a_k$.

We treat all 6 instances in a systematic way:

We write

$$
\frac{z F(z)}{G(z)} = \frac{z}{N_0} = \frac{z}{a_1 + \frac{z}{N_1}} = \frac{z}{a_1 + \frac{z}{a_2 + \frac{z}{N_2}}} = \ldots ,
$$

and set

$$N_i = \frac{r_i}{s_i} .$$

This means

$$N_i = a_{i+1} + \frac{z}{N_{i+1}}$$
or

$$\frac{z}{N_{i+1}} = \frac{z s_{i+1}}{r_{i+1}} = N_i - a_{i+1} = \frac{r_i}{s_i} - a_{i+1} = \frac{r_i - a_{i+1} s_i}{s_i} .$$

We can set $r_i = s_{i-1}$ and get the recursion

$$s_{i+1} z = s_{i-1} - a_{i+1} s_i .$$

The initial conditions are

$$s_{-1} = G(z) \quad \text{and} \quad s_0 = F(z) .$$

Note that the $a_i$’s are the unique numbers that make the $s_i$’s power series expansions.

In all instances, we are able to guess the numbers $a_k$ and the power series $s_k$, and prove the guessed form by induction. In the cotangent case, $F$ and $G$ switch roles, of course. The proof by induction is a routine computation; the challenging part in this line of research is the guessing. Since the proofs are very similar, we present just one of them.

Replacing $q$ by $1/q$ in

$$F(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n+1]_q} q^{An^2+Bn}$$
leads to $F(z)$ with new parameters $A' = 2 - A$ and $B' = 1 - B$. Similarly, $G(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n]_q} q^{Cn^2+Dn}$ leads to new parameters $C' = 2 - C$ and $D' = -1 - D$. So, if there exists a “nice” expansion for $(A, B, C, D)$, there is also one for $(2 - A, 1 - B, 2 - C, -1 - D)$. Slightly more general, replacing $z$ by $z q^\ell$, also for $(2 - A, 1 - B + \ell, 2 - C, -1 - D + \ell)$.

I only found two more expansions which are “nice” and do not follow from the ones already presented in the way just described: The tangent for parameters $(0, 0, 0, 2)$ and the cotangent for $(0, 2, 0, 0)$.

Finally, the special shape of the quantities $s_i$ indicates some candidates for a generalization. Indeed, the last section sketches functions $F_h(z)$ and $G_h(z)$, such that $F_0(z) = F(z)$ and $G_0(z) = G(z)$, with a “nice” continued fraction expansion of $F_h(z)/G_h(z)$ and $G_h(z)/F_h(z)$.

2. TANGENT

2.1. $d = 0$.

$$a_k = (-1)^{k-1} \frac{[2k-1]_q}{q^{k-1}} ,$$
\[ s_k = (-1)^{\frac{k+1}{2}} q^{(k+1)} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 2k + 1]_q} \prod_{j=1}^{k} [2n + 2j]_q. \]

2.2. \( d = 1 \).

\[ a_{2k} = -\frac{[4k - 1]_q}{q^{(k+1)(2k-1)}}, \]
\[ a_{2k+1} = [4k + 1]_q q^{(2k-1)}. \]

\[ s_{2k} = (-1)^k q^{2k} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 1]_q} \prod_{j=1}^{2k} [2n + 2j]_q q^{n(2n+2k)}, \]
\[ s_{2k+1} = (-1)^{k-1} q^{(k+1)(3k+2)} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 3]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q q^{n(2n+2k+2)}. \]

2.3. \( d = 2 \).

\[ a_{2k} = -\frac{[4k - 1]_q (1 - q^{2k} - q^{2k+1} + q^{4k-1})^2}{(1 - q^2)^2 q^{6k-3}}, \]
\[ a_{2k+1} = \frac{[4k + 1]_q (1 - q^2)^2 q^{2k-1}}{(1 - q^{2k+2} - q^{2k+3} + q^{4k+3})(1 - q^{2k} - q^{2k+1} + q^{4k-1})}. \]

\[ s_{2k} = (-1)^k q^{2k^2} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 1]_q} \prod_{j=1}^{2k} [2n + 2j]_q \left(1 + \frac{q^{2n+2}(1 - q^2)(1 - q^{2k+1})}{1 - q^2}\right) q^{2n(2n+2k)}, \]
\[ s_{2k+1} = (-1)^{k-1} q^{2k^2+6k+3} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 3]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q \frac{(1 - q^2)^2 q^{2n+2k+2}}{1 - q^{2k+2} - q^{2k+3} + q^{4k+3}}. \]

3. Cotangent

3.1. \( d = 0 \). \( a_1 = 1 \), and for \( k \geq 1 \)

\[ a_{2k} = \frac{[4k - 1]_q [2k - 1]_q [2k]_q^2}{q^{6k-5}(1 + q)^2}, \]
\[ a_{2k+1} = -\frac{[4k + 1]_q (1 + q)^2 q^{2k-2}}{[2k - 1]_q [2k]_q [2k+1]_q [2k+2]_q}. \]

\[ s_{2k} = (-1)^k q^{k(2k-1)} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 1]_q} \prod_{j=1}^{2k} [2n + 2j]_q \left( [2n + 4k + 1]_q + \frac{q^2[2k]_q [2k - 1]_q}{1 + q}\right), \]
\[ s_{2k+1} = (-1)^{k-1} q^{2k^2+5k+1} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 3]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q \frac{1 + q}{[2k + 1]_q [2k + 2]_q}. \]
3.2. \( d = 1 \). \( a_1 = 1 \), and for \( k \geq 1 \)

\[
\alpha_{2k} = \frac{[4k - 1]_q [k(2k - 1)]^2}{q^{(2k-1)(k+1)}},
\]

\[
\alpha_{2k+1} = -\frac{[4k + 1]_q q^{k(2k-1)}}{[k(2k - 1)]_q [(k+1)(2k+1)]_q}.
\]

\[
s_{2k} = (-1)^k q^{k^2} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 1]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q [2n + 2k^2 + 3k + 1]_q q^{n(2n+2k)},
\]

\[
s_{2k+1} = (-1)^k q^{(k+1)(3k+2)} \sum_{n \geq 0} \frac{z^n(-1)^n}{[(k+1)(2k+1)]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q q^{n(2n+2k+2)}.
\]

3.3. \( d = 2 \). \( a_1 = 1 \), and for \( k \geq 1 \)

\[
\alpha_{2k} = \frac{[4k - 1]_q [2k - 1]_q [2k]_q}{q^{6k-3}(1 + q)^2},
\]

\[
\alpha_{2k+1} = -\frac{[4k + 1]_q (1 + q)^2 q^{2k-1}}{[2k - 1]_q [2k]_q [2k + 1]_q [2k + 2]_q}.
\]

\[
s_{2k} = (-1)^k q^{2k^2} \sum_{n \geq 0} \frac{z^n(-1)^n}{[2n + 4k + 1]_q} \prod_{j=1}^{2k} [2n + 2j]_q q^{2n(2n+2k)} \times
\]

\[
\times \left( [2n + 4k + 1]_q + \frac{q^{2n+2} [2k]_q [2k - 1]_q}{1 + q} \right),
\]

\[
s_{2k+1} = (-1)^k q^{2k^2 + 6k + 3} (1 + q) \sum_{n \geq 0} \frac{z^n(-1)^n}{[2k + 2]_q [2k + 1]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q q^{2n(2n+2k+2)}.
\]

4. Proof of the cotangent case \( d = 1 \)

We have by inspection that \( s_0 = G(z) \), and compute

\[
s_1 = \frac{1}{z} (s_{-1} - s_0)
\]

\[
= \frac{1}{z} \left( F(z) - G(z) \right)
\]

\[
= \frac{1}{z} \sum_{n \geq 0} \frac{z^n(-1)^n q^n}{[2n + 1]_q} \frac{1 - q - 1 + q^{2n+1}}{1 - q}
\]

\[
= \sum_{n \geq 1} \frac{z^{n-1}(-1)^n q^n}{[2n + 1]_q} \frac{-q(1 - q^{2n})}{1 - q}
\]

\[
= \sum_{n \geq 1} \frac{z^{n-1}(-1)^{n-1} q^{n^2+1}}{[2n - 1]_q ![2n + 1]_q}
\]

\[
= q^2 \sum_{n \geq 0} \frac{z^n(-1)^n q^{n(n+2)}}{[2n + 1]_q ![2n + 3]_q}.
\]
which checks, so we have the basis for our induction. And now we must show for all $n$ that

$$[z^n](s_{2k} - a_{2k+2}s_{2k+1}) = [z^{n-1}]s_{2k+2},$$

$$[z^n](s_{2k-1} - a_{2k+1}s_{2k}) = [z^{n-1}]s_{2k+1}.$$  

Let us start with the first one:

$$[z^n](s_{2k} - a_{2k+2}s_{2k+1}) = (-1)^k q^k \frac{(-1)^n}{[2n + 4k + 1]q!} \prod_{j=1}^{2k} [2n + 2j]q[2n + 2k^2 + 3k + 1]q^{n+2k}$$

$$- \frac{[4k + 3]q((k + 1)(2k + 1))^2}{q^{(2k+1)(k+2)}} \times$$

$$\times (1) q^{k(3k+2)} \frac{(-1)^n}{[(k + 1)(2k + 1)]q} \prod_{j=1}^{2k+1} [2n + 2j]q^{n+2k+2}$$

$$= \frac{(-1)^n k^2q^{n+k}q^{(n+k)^2}}{[2n + 4k + 1]q!} \prod_{j=1}^{2k} [2n + 2j]q \times$$

$$\times \left( [2n + 4k + 3]q[2n + 2k^2 + 3k + 1]q - q^{2n}q^{(n+k)^2} \right)$$

$$= \frac{(-1)^n k^2q^{n+k}q^{(n+k)^2}}{[2n + 4k + 1]q!} \prod_{j=1}^{2k} [2n + 2j]q[2n + 2k^2 + 7k + 4]q.$$  

On the other hand

$$[z^{n-1}]s_{2k+2} = (-1)^{k-1} q^{(k+1)^2} \frac{(-1)^{n-1}}{[2n + 4k + 3]q!} \times$$

$$\times \prod_{j=1}^{2k+2} [2n + 2j]q[2n + 2k^2 + 7k + 4]q^{n+1}(n+2k+1),$$  

which is the same, as it should.

And now to the second one:

$$[z^n](s_{2k-1} - a_{2k+1}s_{2k}) = \frac{(-1)^{k-1} q^{k(3k-1)}}{[k(2k - 1)]q} \frac{(-1)^n}{[2n + 4k - 1]q!} \prod_{j=1}^{2k-1} [2n + 2j]q^{n+2k}$$

$$+ \frac{[4k + 1]q^{k(2k-1)}}{[k(2k - 1)]q[(k + 1)(2k + 1)]q} \times$$

$$\times (-1)^n q^{k^2} \frac{(-1)^n}{[2n + 4k + 1]q} \prod_{j=1}^{2k} [2n + 2j]q[2n + 2k^2 + 3k + 1]q^{n+2k}$$
which is the same, so that our proof is finished.

On the other hand,

\[
[z^{n-1}]s_{2k+1} = \frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q [2n + 4k + 1]_q} \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n + 2]_q!} \prod_{j=0}^{2k+1} [2n - 2 + 2j]_q q^{(n-1)(n+2k+1)},
\]

which is the same, so that our proof is finished.

5. A TANGENT

\[
F(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n + 1]_q!},
\]

\[
G(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n]_q!} q^{2n}.
\]

\[
\alpha_{2k} = -\frac{[4k - 1]_q q^{2k-3}(1 - q^2)^2}{(1 - q^{2k-3} - q^{2k-2} + q^{4k-3})(1 - q^{2k-1} - q^{2k} + q^{4k+1})},
\]

\[
\alpha_{2k+1} = \frac{[4k + 1]_q (1 - q^{2k-1} - q^{2k} + q^{4k+1})^2}{q^{6k-2}(1 - q^2)^2}.
\]

\[
s_{2k} = \frac{(-1)^{k-1} q^{2k^2 + 3k - 1} (1 - q^2)}{1 - q^{2k-1} - q^{2k} + q^{4k+1}} \sum_{n \geq 0} \frac{z^n (-1)^n}{[2n + 4k + 1]_q! [2n]_q} \prod_{j=1}^{2k} [2n + 2j]_q,
\]

\[
s_{2k+1} = \frac{(-1)^{k-1} q^{2k^2 + k}}{1 - q^{2k-1} - q^{2k} + q^{4k+1}} \sum_{n \geq 0} \frac{z^n (-1)^n}{[2n + 4k + 3]_q! [2n]_q} \prod_{j=1}^{2k+1} [2n + 2j]_q \times
\]

\[
\times \left( q^{4k+2n+3} - \frac{(1 - q^{2k+1})(1 - q^{2k+2})}{1 - q^2} \right).
\]

6. A COTANGENT

\[
F(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n + 1]_q!} q^{2n},
\]
CONTINUED FRACTION EXPANSIONS

\[ G(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n]_q!}. \]

For \( k \geq 0 \),

\[ a_{2k} = \frac{[4k - 1]_q [2k - 1]_q^2 [2k]_q^2}{q^{6k-6} (1 + q)^2}, \]
\[ a_{2k+1} = -\frac{[4k + 1]_q (1 + q)^2 q^{2k-3}}{[2k - 1]_q [2k]_q [2k + 1]_q [2k + 2]_q}, \]

and \( a_1 = 1/(1 - q) \).

\[ s_{2k} = \frac{(-1)^k q^{2k^2 - k}}{1 - q} \sum_{n \geq 0} \frac{z^n (-1)^n}{[2n + 4k + 1]_q!} \prod_{j=1}^{2k} [2n + 2j]_q \times \left( \frac{1 - q^{2k+1} - q^{2k+2} + q^{4k+1}}{1 - q^2} - q^{2n+4k+1} \right), \]
\[ s_{2k+1} = \frac{(-1)^k q^{2k^2 + 5k} (1 + q)}{[2k + 2]_q [2k + 1]_q} \sum_{n \geq 0} \frac{z^n (-1)^n}{[2n + 4k + 3]_q!} \prod_{j=1}^{2k+1} [2n + 2j]_q. \]

7. A generalization

\[ F_h(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n + 1]_q!} \prod_{j=1}^{h} [2n + 2j]_q q^{dn^2}, \]
\[ G_h(z) = \sum_{n \geq 0} \frac{(-1)^n z^n}{[2n]_q!} \prod_{j=1}^{h} [2n + 2j]_q q^{dn^2}. \]

7.1. \( d = 0 \).

\[ a_{2k} = -[4k - 1 + 2h]_q q^{-2k+1-2h}, \]
\[ a_{2k+1} = [4k + 1 + 2h]_q q^{-2k}. \]

7.2. \( d = 1 \).

\[ a_{2k} = -[4k - 1 + 2h]_q q^{-2k^2 - k(2h+1)+1}, \]
\[ a_{2k+1} = [4k + 1 + 2h]_q q^{2k^2 + k(2h-1)}. \]

7.3. \( d = 2 \).

\[ a_{2k} = -\frac{[4k - 1 + 2h]_q ([k]_q^2 - q^{2k+1+2h} [k - 1]_q^2)^2}{q^{6k-3+2h}}, \]
\[ a_{2k+1} = \frac{[4k + 1 + 2h]_q q^{2k-1+2h}}{([k]_q^2 - q^{2k+1+2h} [k - 1]_q^2)([k + 1]_q^2 - q^{2k+3+2h} [k]_q^2)}. \]
7.4. \( d = 0 \). \( a_1 = 1 \), and for \( k \geq 1 \)

\[
a_{2k} = \frac{[4k - 1 + 2h]_q [2k - 1 + 2h]_q [2k]_q^2}{q^{6k-5+2h}(1 + q)^2},
\]

\[
a_{2k+1} = -\frac{[4k + 1 + 2h]_q (1 + q)^2 q^{2k-2}}{[2k - 1 + 2h]_q [2k]_q [2k + 1 + 2h]_q [2k + 2]_q}.
\]

7.5. \( d = 1 \). \( a_1 = 1 \), and for \( k \geq 1 \)

\[
a_{2k} = \frac{[4k - 1 + 2h]_q [k(2k - 1 + 2h)]_q^2}{q^{(2k-1)(k+1)+2kh}},
\]

\[
a_{2k+1} = -\frac{[4k + 1 + 2h]_q q^{k(2k-1)+2kh}}{[k(2k - 1 + 2h)]_q [(k + 1)(2k + 1 + 2h)]_q}.
\]

7.6. \( d = 2 \). \( a_1 = 1 \), and for \( k \geq 1 \)

\[
a_{2k} = \frac{[4k - 1 + 2h]_q [2k - 1 + 2h]_q [2k]_q^2}{q^{6k-3+2h}(1 + q)^2},
\]

\[
a_{2k+1} = -\frac{[4k + 1 + 2h]_q (1 + q)^2 q^{2k-1+2h}}{[2k - 1 + 2h]_q [2k]_q [2k + 1 + 2h]_q [2k + 2]_q}.
\]

Acknowledgment. Thanks are due to Ae Ja Yee for valuable discussions.

References


Helmut Prodinger, Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa

E-mail address: hproding@sun.ac.za