CONTINUED FRACTION EXPANSIONS FOR q-TANGENT AND q-COTANGENT FUNCTIONS

HELMUT PRODINGER

Abstract. For 3 different versions of q-tangent resp. q-cotangent functions, we compute the continued fraction expansion explicitly, by guessing the relative quantities and proving the recursive relation afterwards. It is likely that these are the only instances with a "nice" expansion.

1. Introduction

In this paper, we consider the functions

$$F(z) = \sum_{n\ge 0} \frac{(-1)^n z^n}{[2n+1]_q!} q^{dn^2},$$

$$G(z) = \sum_{n\ge 0} \frac{(-1)^n z^n}{[2n]_q!} q^{dn^2}.$$

We use standard q-notation:

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := [1]_q[2]_q \dots [n]_q.$$

For $d = 0, 1, 2$, we will find the following continued fraction expansions:

$$\frac{zF(z)}{G(z)} = \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cfrac{z}{\ddots}}}}$$

(Replacing z by z^2, we get z times a q-tangent function.)

$$\frac{zG(z)}{F(z)} = \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cfrac{z}{\ddots}}}}$$

(Replacing z by z^2, we get z^3 times a q-cotangent function.)

These q-trigonometric functions are variants of Jackson's, see [2].

The instance $d = 0$ of the q-tangent appeared in [3], and the instance $d = 1$ in [1] and [4]. The instance $d = 2$ as well as the q-cotangent expansions seem to be new.

Date: June 12, 2008.

Key words and phrases. q-tangent, q-cotangent, continued fraction.

Computer experiments indicate that, apart from trivial variations, these are the only cases where we get "nice" coefficients a_k.

We treat all 6 instances in a systematic way:

We write

$$\frac{zF(z)}{G(z)} = \frac{z}{N_0} = \cfrac{z}{a_1 + \cfrac{z}{N_1}} = \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{N_2}}} = \dots,$$

and set

$$N_i = \frac{r_i}{s_i}.$$

This means

$$N_i = a_{i+1} + \frac{z}{N_{i+1}}$$

or

$$\frac{z}{N_{i+1}} = \frac{zs_{i+1}}{r_{i+1}} = N_i - a_{i+1} = \frac{r_i}{s_i} - a_{i+1} = \frac{r_i - a_{i+1}s_i}{s_i}.$$

We can set $r_i = s_{i-1}$ and get the recursion

$$s_{i+1}z = s_{i-1} - a_{i+1}s_i.$$

The initial conditions are

$$s_{-1} = G(z) \quad \text{and} \quad s_0 = F(z).$$

Note that the a_i's are the unique numbers that make the s_i's power series expansions.

In all instances, we are able to guess the numbers a_k and the power series s_k, and prove the guessed form by induction. In the cotangent case, F and G switch roles, of course. The proof by induction is a routine computation; the challenging part in this line of research is the guessing. Since the proofs are very similar, we present just one of them.

Replacing q by $1/q$ in $F(z) = \sum_{n\ge0} \frac{(-1)^n z^n}{[2n+1]_q!} q^{An^2+Bn}$ leads to $F(z)$ with new parameters $A' = 2 - A$ and $B' = 1 - B$. Similarly, $G(z) = \sum_{n\ge0} \frac{(-1)^n z^n}{[2n]_q!} q^{Cn^2+Dn}$ leads to new parameters $C' = 2 - C$ and $D' = -1 - D$. So, if there exists a "nice" expansion for (A, B, C, D), there is also one for $(2 - A, 1 - B, 2 - C, -1 - D)$. Slightly more general, replacing z by zq^ℓ, also for $(2 - A, 1 - B + \ell, 2 - C, -1 - D + \ell)$.

I only found two more expansions which are "nice" and do not follow from the ones already presented in the way just described: The tangent for parameters $(0, 0, 0, 2)$ and the cotangent for $(0, 2, 0, 0)$.

Finally, the special shape of the quantities s_i indicates some candidates for a generalization. Indeed, the last section sketches functions $F_h(z)$ and $G_h(z)$, such that $F_0(z) = F(z)$ and $G_0(z) = G(z)$, with a "nice" continued fraction expansion of $F_h(z)/G_h(z)$ and $G_h(z)/F_h(z)$.

2. Tangent

2.1. $d = 0$.

$$a_k = (-1)^{k-1} \frac{[2k-1]_q}{q^{k-1}},$$

$$s_k = (-1)^{\lfloor \frac{k+1}{2} \rfloor} q^{\binom{k+1}{2}} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+2k+1]_q!} \prod_{j=1}^{k} [2n+2j]_q.$$

2.2. $d = 1$.

$$a_{2k} = -\frac{[4k-1]_q}{q^{(k+1)(2k-1)}},$$
$$a_{2k+1} = [4k+1]_q q^{k(2k-1)}.$$

$$s_{2k} = (-1)^k q^{k^2} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q q^{n(n+2k)},$$
$$s_{2k+1} = (-1)^{k-1} q^{(k+1)(3k+2)} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q q^{n(n+2k+2)}.$$

2.3. $d = 2$.

$$a_{2k} = -\frac{[4k-1]_q(1-q^{2k}-q^{2k+1}+q^{4k-1})^2}{(1-q^2)^2 q^{6k-3}},$$
$$a_{2k+1} = \frac{[4k+1]_q(1-q^2)^2 q^{2k-1}}{(1-q^{2k+2}-q^{2k+3}+q^{4k+3})(1-q^{2k}-q^{2k+1}+q^{4k-1})}.$$

$$s_{2k} = (-1)^k q^{2k^2} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q \Big(1 + \frac{q^{2n+2}(1-q^{2k})(1-q^{2k+1})}{1-q^2}\Big) q^{2n(n+2k)},$$
$$s_{2k+1} = (-1)^{k-1} q^{2k^2+6k+3} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q \frac{(1-q^2)q^{2n(n+2k+2)}}{1-q^{2k+2}-q^{2k+3}+q^{4k+3}}.$$

3. Cotangent

3.1. $d = 0$. $a_1 = 1$, and for $k \geq 1$

$$a_{2k} = \frac{[4k-1]_q[2k-1]_q^2[2k]_q^2}{q^{6k-5}(1+q)^2},$$
$$a_{2k+1} = -\frac{[4k+1]_q(1+q)^2 q^{2k-2}}{[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q}.$$

$$s_{2k} = (-1)^k q^{k(2k-1)} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q \left([2n+4k+1]_q + \frac{q^2[2k]_q[2k-1]_q}{1+q}\right),$$
$$s_{2k+1} = (-1)^k q^{2k^2+5k+1} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q \frac{1+q}{[2k+1]_q[2k+2]_q}.$$

3.2. $d=1$. $a_1=1$, and for $k\ge 1$

$$a_{2k}=\frac{[4k-1]_q[k(2k-1)]_q^2}{q^{(2k-1)(k+1)}},$$
$$a_{2k+1}=-\frac{[4k+1]_q q^{k(2k-1)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q}.$$

$$s_{2k}=(-1)^k q^{k^2}\sum_{n\ge 0}\frac{z^n(-1)^n}{[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q[2n+2k^2+3k+1]_q q^{n(n+2k)},$$
$$s_{2k+1}=\frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q}\sum_{n\ge 0}\frac{z^n(-1)^n}{[2n+4k+3]_q!}\prod_{j=1}^{2k+1}[2n+2j]_q q^{n(n+2k+2)}.$$

3.3. $d=2$. $a_1=1$, and for $k\ge 1$

$$a_{2k}=\frac{[4k-1]_q[2k-1]_q^2[2k]_q^2}{q^{6k-3}(1+q)^2},$$
$$a_{2k+1}=-\frac{[4k+1]_q(1+q)^2q^{2k-1}}{[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q}.$$

$$s_{2k}=(-1)^k q^{2k^2}\sum_{n\ge 0}\frac{z^n(-1)^n}{[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q q^{2n(n+2k)}\times$$
$$\times\left([2n+4k+1]_q+\frac{q^{2n+2}[2k]_q[2k-1]_q}{1+q}\right),$$
$$s_{2k+1}=\frac{(-1)^k q^{2k^2+6k+3}(1+q)}{[2k+2]_q[2k+1]_q}\sum_{n\ge 0}\frac{z^n(-1)^n}{[2n+4k+3]_q!}\prod_{j=1}^{2k+1}[2n+2j]_q q^{2n(n+2k+2)}.$$

4. Proof of the cotangent case $d=1$

We have by inspection that $s_0=G(z)$, and compute

$$\begin{aligned}
s_1&=\frac{1}{z}\big(s_{-1}-s_0\big)\\
&=\frac{1}{z}\big(F(z)-G(z)\big)\\
&=\frac{1}{z}\sum_{n\ge 0}\frac{z^n(-1)^n q^{n^2}}{[2n+1]_q!}\frac{1-q-1+q^{2n+1}}{1-q}\\
&=\sum_{n\ge 1}\frac{z^{n-1}(-1)^n q^{n^2}}{[2n+1]_q!}\frac{-q(1-q^{2n})}{1-q}\\
&=\sum_{n\ge 1}\frac{z^{n-1}(-1)^{n-1}q^{n^2+1}}{[2n-1]_q![2n+1]_q}\\
&=q^2\sum_{n\ge 0}\frac{z^n(-1)^n q^{n(n+2)}}{[2n+1]_q![2n+3]_q},
\end{aligned}$$

which checks, so we have the basis for our induction. And now we must show for all n that

$$[z^n]\big(s_{2k} - a_{2k+2}s_{2k+1}\big) = [z^{n-1}]s_{2k+2},$$
$$[z^n]\big(s_{2k-1} - a_{2k+1}s_{2k}\big) = [z^{n-1}]s_{2k+1}.$$

Let us start with the first one:

$$\begin{aligned}
[z^n]\big(s_{2k} - a_{2k+2}s_{2k+1}\big) &= (-1)^k q^{k^2} \frac{(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q [2n+2k^2+3k+1]_q q^{n(n+2k)} \\
&- \frac{[4k+3]_q[(k+1)(2k+1)]_q^2}{q^{(2k+1)(k+2)}} \times \\
&\times \frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q} \frac{(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q q^{n(n+2k+2)} \\
&= \frac{(-1)^{n+k} q^{k^2}}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q [2n+2k^2+3k+1]_q q^{n(n+2k)} \\
&- \frac{(-1)^{n+k} q^{k^2} [4k+3]_q [(k+1)(2k+1)]_q}{[2n+4k+1]_q! [2n+4k+3]_q} \prod_{j=1}^{2k} [2n+2j]_q q^{n(n+2k+2)} \\
&= \frac{(-1)^{n+k} q^{(n+k)^2}}{[2n+4k+1]_q! [2n+4k+3]_q} \prod_{j=1}^{2k} [2n+2j]_q \times \\
&\quad \times \Big([2n+4k+3]_q [2n+2k^2+3k+1]_q - q^{2n} [4k+3]_q [(k+1)(2k+1)]_q \Big) \\
&= \frac{(-1)^{n+k} q^{(n+k)^2}}{[2n+4k+1]_q! [2n+4k+3]_q} \prod_{j=1}^{2k} [2n+2j]_q [2n]_q [2n+2k^2+7k+4]_q.
\end{aligned}$$

On the other hand

$$\begin{aligned}
[z^{n-1}]s_{2k+2} &= (-1)^{k-1} q^{(k+1)^2} \frac{(-1)^{n-1}}{[2n+4k+3]_q!} \times \\
&\times \prod_{j=1}^{2k+2} [2n-2+2j]_q [2n+2k^2+7k+4]_q q^{(n-1)(n+2k+1)},
\end{aligned}$$

which is the same, as it should.

And now to the second one:

$$\begin{aligned}
[z^n]\big(s_{2k-1} - a_{2k+1}s_{2k}\big) &= \frac{(-1)^{k-1} q^{k(3k-1)}}{[k(2k-1)]_q} \frac{(-1)^n}{[2n+4k-1]_q!} \prod_{j=1}^{2k-1} [2n+2j]_q q^{n(n+2k)} \\
&+ \frac{[4k+1]_q q^{k(2k-1)}}{[k(2k-1)]_q [(k+1)(2k+1)]_q} \times \\
&\times (-1)^k q^{k^2} \frac{(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q [2n+2k^2+3k+1]_q q^{n(n+2k)}
\end{aligned}$$

$$= \frac{(-1)^{n+k-1}q^{k(3k-1)+n(n+2k)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q\times$$
$$\times\Bigl([2n+4k+1]_q[(k+1)(2k+1)]_q-[4k+1]_q[2n+2k^2+3k+1]_q\Bigr)$$
$$= \frac{(-1)^{n+k-1}q^{k(3k-1)+n(n+2k)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q[2n]_q[k(2k-1)]_q q^{4k+1}$$
$$= \frac{(-1)^{n+k-1}q^{3k(k+1)+1+n(n+2k)}}{[(k+1)(2k+1)]_q[2n+4k+1]_q!}\prod_{j=0}^{2k}[2n+2j]_q.$$

On the other hand,

$$[z^{n-1}]s_{2k+1} = \frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q}\frac{(-1)^{n-1}}{[2n+4k+1]_q!}\prod_{j=1}^{2k+1}[2n-2+2j]_q q^{(n-1)(n+2k+1)},$$

which is the same, so that our proof is finished.

5. A tangent

$$F(z) = \sum_{n\ge0}\frac{(-1)^n z^n}{[2n+1]_q!},$$
$$G(z) = \sum_{n\ge0}\frac{(-1)^n z^n}{[2n]_q!}q^{2n}.$$

$$a_{2k} = -\frac{[4k-1]_q q^{2k-3}(1-q^2)^2}{(1-q^{2k-3}-q^{2k-2}+q^{4k-3})(1-q^{2k-1}-q^{2k}+q^{4k+1})},$$
$$a_{2k+1} = \frac{[4k+1]_q(1-q^{2k-1}-q^{2k}+q^{4k+1})^2}{q^{6k-2}(1-q^2)^2}.$$

$$s_{2k} = \frac{(-1)^{k-1}q^{2k^2+3k-1}(1-q^2)}{1-q^{2k-1}-q^{2k}+q^{4k+1}}\sum_{n\ge0}\frac{z^n(-1)^n}{[2n+4k+1]_q![2n]_q}\prod_{j=1}^{2k}[2n+2j]_q,$$
$$s_{2k+1} = (-1)^{k-1}q^{2k^2+k}\sum_{n\ge0}\frac{z^n(-1)^n}{[2n+4k+3]_q![2n]_q}\prod_{j=1}^{2k+1}[2n+2j]_q\times$$
$$\times\Bigl(q^{4k+2n+3}-\frac{(1-q^{2k+1})(1-q^{2k+2})}{1-q^2}\Bigr).$$

6. A cotangent

$$F(z) = \sum_{n\ge0}\frac{(-1)^n z^n}{[2n+1]_q!}q^{2n},$$

$$G(z)=\sum_{n\geq 0}\frac{(-1)^n z^n}{[2n]_q!}.$$

For $k\geq 0$,

$$a_{2k}=\frac{[4k-1]_q[2k-1]_q^2[2k]_q^2}{q^{6k-6}(1+q)^2},$$
$$a_{2k+1}=-\frac{[4k+1]_q(1+q)^2q^{2k-3}}{[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q},$$

and $a_1=1/(1-q)$.

$$s_{2k}=\frac{(-1)^k q^{2k^2-k}}{1-q}\sum_{n\geq 0}\frac{z^n(-1)^n}{[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q\times$$
$$\times\Big(\frac{1-q^{2k+1}-q^{2k+2}+q^{4k+1}}{1-q^2}-q^{2n+4k+1}\Big),$$
$$s_{2k+1}=\frac{(-1)^k q^{2k^2+5k}(1+q)}{[2k+2]_q[2k+1]_q}\sum_{n\geq 0}\frac{z^n(-1)^n}{[2n+4k+3]_q!}\prod_{j=1}^{2k+1}[2n+2j]_q.$$

7. A generalization

$$F_h(z)=\sum_{n\geq 0}\frac{(-1)^n z^n}{[2n+1]_q!}\prod_{j=1}^{h}[2n+2j]_q q^{dn^2},$$
$$G_h(z)=\sum_{n\geq 0}\frac{(-1)^n z^n}{[2n]_q!}\prod_{j=1}^{h}[2n+2j]_q q^{dn^2}.$$

7.1. $d=0$.

$$a_{2k}=-[4k-1+2h]_q q^{-2k+1-2h},$$
$$a_{2k+1}=[4k+1+2h]_q q^{-2k}.$$

7.2. $d=1$.

$$a_{2k}=-[4k-1+2h]_q q^{-2k^2-k(2h+1)+1},$$
$$a_{2k+1}=[4k+1+2h]_q q^{2k^2+k(2h-1)}.$$

7.3. $d=2$.

$$a_{2k}=-\frac{[4k-1+2h]_q([k]_{q^2}-q^{2k+1+2h}[k-1]_{q^2})^2}{q^{6k-3+2h}},$$
$$a_{2k+1}=\frac{[4k+1+2h]_q q^{2k-1+2h}}{([k]_{q^2}-q^{2k+1+2h}[k-1]_{q^2})([k+1]_{q^2}-q^{2k+3+2h}[k]_{q^2})}.$$

7.4. $d=0$. $a_1=1$, and for $k\ge 1$

$$a_{2k}=\frac{[4k-1+2h]_q[2k-1+2h]_q^2[2k]_q^2}{q^{6k-5+2h}(1+q)^2},$$
$$a_{2k+1}=-\frac{[4k+1+2h]_q(1+q)^2q^{2k-2}}{[2k-1+2h]_q[2k]_q[2k+1+2h]_q[2k+2]_q}.$$

7.5. $d=1$. $a_1=1$, and for $k\ge 1$

$$a_{2k}=\frac{[4k-1+2h]_q[k(2k-1+2h)]_q^2}{q^{(2k-1)(k+1)+2kh}},$$
$$a_{2k+1}=-\frac{[4k+1+2h]_q q^{k(2k-1)+2kh}}{[k(2k-1+2h)]_q[(k+1)(2k+1+2h)]_q}.$$

7.6. $d=2$. $a_1=1$, and for $k\ge 1$

$$a_{2k}=\frac{[4k-1+2h]_q[2k-1+2h]_q^2[2k]_q^2}{q^{6k-3+2h}(1+q)^2},$$
$$a_{2k+1}=-\frac{[4k+1+2h]_q(1+q)^2q^{2k-1+2h}}{[2k-1+2h]_q[2k]_q[2k+1+2h]_q[2k+2]_q}.$$

Acknowledgment. Thanks are due to Ae Ja Yee for valuable discussions.

References

[1] M. Fulmek, A continued fraction expansion for a q-tangent function, Sém. Loth. Comb., **45**, (2000), [B45b] (5 pages).

[2] F.H. Jackson, A basic-sine and cosine with symbolic solutions of certain differential equations, Proc. Edinburgh Math. Soc., **22**, (1904), 28–39.

[3] H. Prodinger, *Combinatorics of geometrically distributed random variables: New q-tangent and q-secant numbers*, Int. J. Math. Math. Sci., **24**, (2000), 825–838.

[4] H. Prodinger, *A continued fraction expansion for a q-tangent function: An elementary proof*, Sém. Loth. Comb., **60**, (2008), [B60b] (3 pages).

Helmut Prodinger, Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa

E-mail address: hproding@sun.ac.za