

# ON IDENTITIES BY LARCOMBE–FENNESSEY AND CASSINI

HELMUT PRODINGER<sup>‡</sup> AND STEPHAN WAGNER\*

ABSTRACT. A recent identity of Larcombe and Fennessey is derived via a weighted version of Cassini's identity for Fibonacci numbers.

## 1. THE IDENTITIES

Let  $M = \begin{pmatrix} V & U \\ W & 0 \end{pmatrix}$  and  $\alpha_n = (1 \ 0) M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

In [2], the non-linear identity

$$(-1)^n U^n W^n = \alpha_n^2 - UW\alpha_{n-1}^2 - V\alpha_n\alpha_{n-1}$$

( $n \geq 1$ ) was presented. Actually, in [2],  $V$  and  $W$  were replaced by  $-V$  and  $-W$ , respectively, and the quantities  $U, V, W$  could depend on a parameter  $x$ .

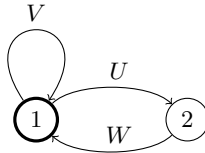
Here, we want to link this identity to the classical Cassini identity

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n$$

for Fibonacci numbers; we will deduce the Larcombe–Fennessey identity from the Cassini identity.

## 2. INTERPRETATION AS WALKS IN A GRAPH

Consider the following graph:



Then  $\alpha_n$  may be interpreted as the sum over all walks of length  $n$  from state 1 to state 1, where each walk is coded by the letter attached to the directed edge. For example,

$$\alpha_4 = V^4 + UWV^2 + UWUW + VUWV + V^2UW.$$

Since a walk can start either with  $V$  or  $UW$ , we have the recursion formula

$$\alpha_n = V\alpha_{n-1} + UW\alpha_{n-2}.$$

This works for  $n \geq 1$ , provided we set  $\alpha_{-1} = 0$ .

Consequently, we have

$$\alpha_{n+1}\alpha_{n-1} = V\alpha_{n-1}\alpha_n + UW\alpha_{n-1}^2.$$

Therefore, the Larcombe–Fennessey identity follows from the simpler identity

$$\alpha_n^2 - \alpha_{n+1}\alpha_{n-1} = (-1)^n U^n W^n.$$

<sup>‡</sup>This author was supported by an incentive grant of the National Research Foundation of South Africa.

\* Supported financially by the National Research Foundation of South Africa under grant number 70560.

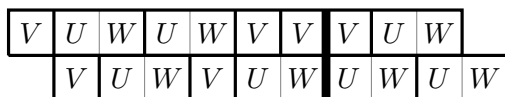
We will deduce this one from Cassini's identity.

### 3. INTERPRETATION AS TILINGS OF AN $n \times 1$ RECTANGLE

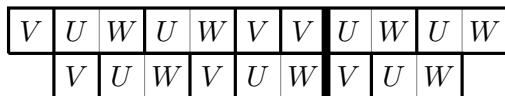
We want to tile an  $n \times 1$  rectangle using  $1 \times 1$  and  $2 \times 1$  rectangles. Each such tiling is in obvious correspondence with a walk, where the edge  $V$  corresponds to a  $1 \times 1$  rectangle, and the two consecutive edges  $UW$  correspond to a  $2 \times 1$  rectangle. For example, the walk  $VUWUWVVUW$  can be interpreted as



It is plain to see, compare [1, p. 1] that the number of tilings of an  $n \times 1$  rectangle is  $F_{n+1}$ , a Fibonacci number. We refer to the graphical proof of Cassini's identity in [1, p. 8] which we repeat here for the readers' convenience. Consider two such tilings, which we arrange in 2 rows, but the second one shifted one unit to the right. Their number is  $F_{n+1}^2$ ; we call this a type 1 tiling. Here is an example:



The rightmost vertical line that is common to both tilings is especially indicated. Now the part to the right of this line will be flipped: top and bottom are exchanged; the result we will call a type 2 tiling:



The number of Type 2 tilings is  $F_{n+2}F_n$ . Note that this operation is reversible, and this mapping is "almost" a bijection. There is a correction to be made, namely when a common vertical line does not exist. Let  $n = 2m$  be even. Then there is a tiling of the first type, namely both rows are  $(UW)^m$ , which has no correspondence of the second type. On the other hand, if  $n = 2m + 1$  is odd, there is a tiling of the second type, namely  $(UW)^{m+1}$  in the first row and  $(UW)^m$  in the second row, which has no corresponding element of the first type. In [1, p. 8], this is only used for the numbers of tilings, but the operation is *weight preserving*. Putting things together, we have shown that

$$\alpha_n^2 - \alpha_{n-1}\alpha_{n+1} = \begin{cases} (UW)^m(UW)^m & \text{for } n = 2m, \\ -(UW)^{m+1}(UW)^m & \text{for } n = 2m + 1, \end{cases}$$

which is the identity that we needed to prove.

### REFERENCES

- [1] A. T. Benjamin and J. J. Quinn, *Proofs that really count*, The Mathematical Association of America, Washington D. C., 2003.
- [2] P. J. Larcombe and E. J. Fennessey, *A non-linear identity for a particular class of polynomial families*, The Fibonacci Quarterly 52 (2014), 75–79.

MSC2010: 11B37, 05A19

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH 7602, STELLENBOSCH, SOUTH AFRICA  
*E-mail address:* `hproding@sun.ac.za`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH 7602, STELLENBOSCH, SOUTH AFRICA  
*E-mail address:* `swagner@sun.ac.za`