ENUMERATION OF S-MOTZKIN PATHS FROM LEFT TO RIGHT AND FROM RIGHT TO LEFT — A KERNEL METHOD APPROACH

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ABSTRACT. The area of S-Motzkin paths (bijective to ternary trees) is calculated using the kernel method by enumerating these (partial) paths with fixed end-point resp. starting point.

1. INTRODUCTION

The area of a lattice path $\mathscr{A}$ is defined to be $\sum_{i\ge 0} h_i(\mathscr{A})$ where $h_i(\mathscr{A})$ is the height of the path $\mathscr{A}$ at x-coordinate i. This parameter has been studied in various types of paths [8, 9, 1]. In this paper, the area will be available as a corollary of the enumeration of partial (incomplete) families of lattice paths that are bijective to ternary trees, and thus enumerated by the numbers $\frac{1}{2n+1}\binom{3n}{n}$.

We study such a family that was recently introduced by [11], and find explicit formulae for them ending after n steps at level k, both, from left to right (starting at the origin) and also from right to left (starting at the end and going backwards). The latter instance is the more challenging one.

S-Motzkin paths can be transformed into other more traditional ternary objects, like ternary trees and ternary paths. No use of this will, however, made here, to keep the discussion self-contained. This is also beneficial from a pedagogic point of view, since it shows how to deal with a system of two equations and the kernel method, in the presence of cubic equations.

The enumeration in this paper involves a lattice path called an S-Motzkin path. This is a subclass of Motzkin paths introduced by the authors in a previous publication [11]. For completeness, we provide the definition again.

Definition 1. *An* S-Motzkin path *of length* $3n$ *is*

- *a Motzkin path*
- *it consists of n up-steps* $(1,1)$*, n down-steps* $(1,-1)$*, n level-steps* $(1,0)$
- *ignoring the down-steps, the level-steps and the up-steps alternate, starting with a level-step.*

We apply the kernel method to S-Motzkin paths as well as reverse S-Motzkin paths to obtain the area of S-Motzkin paths. As a bonus, we obtain the exact enumeration of partial S-Motzkin paths. It is particularly worthwhile to obtain results for this relatively new class of lattice paths, since, as seen in [7], this family appeared in a surprisingly unrelated context.

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It is perhaps of independent interest to note that S-Motzkin paths were introduced to solve a problem from a student olympiad about frog hops [10].

A state-of-the-art survey about lattice path enumeration is [6]; it does not include S-Motzkin paths, as they are new.

Our main findings are the enumerations of four classes of paths (defined later): (1), (2), (3), (4), and the area (7). Informally, these are the enumeration of (partial) S-Motzkin paths, ending on a prescribed level, both, when considering them from left-to-right and from right-to-left. The area of an S-Motzkin is the sum of all its ordinates. The sum of this is computed, when summing over all S-Motzkin paths of the same length $3n$.

2. Preliminary computations

The following computation appears frequently in this paper, and we want to do it only once: Here, $x = t(1-t)^2$, and in all applications we will have $z^3 = x$.

$$\begin{aligned}[x^m]\frac{1}{(1-t)^j} &= \frac{1}{2\pi i}\oint \frac{dx}{x^{m+1}}\frac{1}{(1-t)^j}\\ &= \frac{1}{2\pi i}\oint \frac{dt(1-3t)(1-t)}{t^{m+1}(1-t)^{2m+2}}\frac{1}{(1-t)^j}\\ &= [t^m]\frac{1-3t}{(1-t)^{2m+j+1}} = \binom{3m+j}{m} - 3\binom{3m+j-1}{m-1}.\end{aligned}$$

The binomial series notation as given in [5]

$$\mathscr{B}_t(x)^r = \sum_{k\ge 0}\binom{tk+r}{k}\frac{r}{tk+r}x^k$$

can often be used to express certain quantities that appear in this paper. This is elegant, but the notation using the variable t (as in $x = t(1-t)^2$) seems to be more efficient. Note that the coefficient of x^n in $\mathscr{B}_3(x)^1$ is $\frac{1}{3n+1}\binom{3n+1}{n} = \frac{1}{2n+1}\binom{3n}{n}$, which is the number of ternary trees of size n.

The generating function for ternary trees and also S-Motzkin paths and ternary paths can be rewritten as $\mathscr{B}_3(x)$ using this notation. We will frequently use the substitution $x = t(1-t)^2$.

From the Lagrange inversion formula [2, Theorem A.2] we find

$$[x^n]t^k = \frac{k}{n}[w^{n-k}]\frac{1}{(1-w)^{2n}} = \frac{k}{n}\binom{3n-k-1}{n-k} \qquad \Rightarrow \qquad t^k = \sum_{n\ge k}\binom{3n-k-1}{n-k}\frac{k}{n}x^n.$$

Using this we can compute an expansion that is useful in the context of ternary paths and variants:

$$\sqrt{4t-3t^2} = 2t^{1/2}\sqrt{1-\tfrac{3}{4}t} = 2t^{1/2}\sum_{k\ge 0}(-1)^k(\tfrac{3}{4})^k\binom{\frac{1}{2}}{k}t^k$$

$$= 2t^{1/2}\sum_{k\ge 0}(-1)^k(\tfrac{3}{4})^k\binom{\frac12}{k}\sum_{n\ge k}\binom{3n-k-1}{n-k}\frac{k}{n}x^n$$
$$= 2t^{1/2}\sum_{n\ge 0}x^n\sum_{1\le k\le n}(-1)^k(\tfrac{3}{4})^k\binom{\frac12}{k}\binom{3n-k-1}{n-k}\frac{k}{n}$$
$$= -2t^{1/2}\sum_{n\ge 0}\binom{3n-\frac32}{2n}\frac{1}{2n-1}x^n.$$

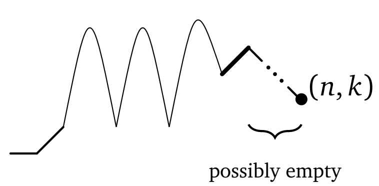
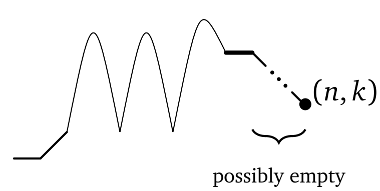
The simplification of the inner sum was done by a computer.

This is one way to switch between expressions in the variable t and expressions in terms of the binomial series notation.

3. The enumeration of partial S-Motzkin paths

3.1. **S-Motzkin paths.** A partial S-Motzkin path is a Motzkin path that can be continued to be an S-Motzkin path, or we might say the first part (of length m, say) of an existing S-Motzkin path. Since S-Motzkin paths are not symmetric w. r. t. left vs. right, we will later also consider this concept from right to left (which turns out to be more difficult).

Let $a_{n,k}$ denote the number of partial S-Motzkin path of length n which ends at height k and the last step of the path from the step set $\{(1,0),(1,1)\}$ is a $(1,1)$ step. Similarly, let $b_{n,k}$ denote the number of partial S-Motzkin path of length n which ends at height k and the last step of the path from the step set $\{(1,0),(1,1)\}$ is a $(1,0)$ step. It is convenient to set $a_{0,0}=1$ and $b_{0,0}=0$.

The path represented by $a_{n,k}$	The path represented by $b_{n,k}$
	

It is easily seen that the following recurrence relations hold:

$$a_{n,k} = b_{n-1,k-1} + a_{n-1,k+1},$$
$$b_{n,k} = a_{n-1,k} + b_{n-1,k+1}.$$

These recurrence relations are valid for $n \ge 1$ and $k \ge 0$; the quantity $b_{n-1,-1}$ must be interpreted as 0. We let

$$A(z,u) = \sum_{n\ge 0}\sum_{k\ge 0} a_{n,k}\, z^n u^k \qquad \text{and} \qquad B(z,u) = \sum_{n\ge 0}\sum_{k\ge 0} b_{n,k}\, z^n u^k$$

and sum over n and k to obtain the following system of equations

$$A(z,u)-1 = zuB(z,u)+\frac{z}{u}A(z,u)-\frac{z}{u}A(z,0),$$
$$B(z,u) = zA(z,u)+\frac{z}{u}B(z,u)-\frac{z}{u}B(z,0).$$

Solving the system of equations for $A(z,u)$ and $B(z,u)$ gives

$$A(z,u)=\frac{-u^2+zuA(z,0)+zu-z^2A(z,0)+B(z,0)z^2u^2}{z^2u^3-u^2+2zu-z^2},$$
$$B(z,u)=\frac{z(-u^2+uB(z,0)+zuA(z,0)-zB(z,0))}{z^2u^3-u^2+2zu-z^2}.$$

The polynomial in the denominator, $z^2u^3-u^2+2zu-z^2$, is of interest to us. Using the substitution $u=zw$ along with $z^3=t(1-t)^2$, we obtain

$$(t^2w^2-2tw^2+tw+w^2-2w+1)(tw-1)=0.$$

Therefore the three roots (expressed again in the variable u) are given by

$$v_1=\frac{z}{t},\qquad v_2=-z\frac{t-2+\sqrt{4t-3t^2}}{2(1-t)^2},\qquad v_3=-z\frac{t-2-\sqrt{4t-3t^2}}{2(1-t)^2}.$$

Alternatively, the three roots can be written using $\mathscr{B}_3(z^3)$ and $\mathscr{B}_{3/2}(\pm z^{3/2})$, but this will not be used.

The roots can also be expressed as series (Puiseux series, to be exact)

$$v_1=z^{-2}-2\sum_{n\ge0}\frac{(3n)!}{(2n)!(n+1)!}z^{3n+1},$$
$$v_2=-6\sum_{n\ge0}\frac{(6n+1)!(n+1)!}{(3n)!(2n+3)!(2n)!2^{4n}}z^{3n+5/2}+\sum_{n\ge0}\frac{(3n)!}{(2n)!(n+1)!}z^{3n+1},$$
$$v_3=6\sum_{n\ge0}\frac{(6n+1)!(n+1)!}{(3n)!(2n+3)!(2n)!2^{4n}}z^{3n+5/2}+\sum_{n\ge0}\frac{(3n)!}{(2n)!(n+1)!}z^{3n+1},$$

and it can be shown that these series are indeed the roots by converting them to hypergeometric functions and using Clausen's identity [12], but we mention this just for interest and will not use it further. Gessel and Xin [3] used such an approach, but it is a tour the force and fortunately, here, we can avoid to go that route.

Note that

$$v_2+v_3=-\frac{z(t-2)}{(t-1)^2}\quad\text{and}\quad v_2v_3=\frac{z^2}{(t-1)^2}.$$

We know that $A(z,u)$ and $B(z,u)$ have power series expansions around $(0,0)$, so the factors $(u-v_2)$ and $(u-v_3)$ in the denominator must also be factors in the numerator. Hence we can find $A(z,0)$ and $B(z,0)$ by solving the system

$$\begin{aligned} 0 &= -v_2^2 + zv_2A(z,0) + zv_2 - z^2A(z,0) + B(z,0)z^2v_2^2,\\ 0 &= -v_3^2 + v_3B(z,0) + zv_3A(z,0) - zB(z,0), \end{aligned}$$

to obtain

$$A(z,0) = -\frac{v_2v_3}{z^2} + \frac{v_2}{z} + \frac{v_3}{z} \qquad \text{and} \qquad B(z,0) = \frac{v_2v_3}{z}.$$

Substituting these back into the original equations yields

$$\begin{aligned} A(z,u) &= \frac{u^2v_2v_3z^2 - uv_2v_3 - u^2z + uv_2z + uv_3z + v_2v_3z + uz^2 - v_2z^2 - v_3z^2}{(u-v_1)(u-v_2)(u-v_3)z^3},\\ B(z,u) &= -\frac{1}{(u-v_1)z}. \end{aligned}$$

Since we know that $(u-v_2)$ and $(u-v_3)$ are factors of the numerator of $A(z,u)$ we can simplify $A(z,u)$ by dividing these two factors out (we consistently use the variable t for that). After simplification,

$$A(z,u) = \frac{1}{1-\frac{tu}{z}} \qquad \text{and} \qquad B(z,u) = \frac{t}{z^2}\frac{1}{1-\frac{tu}{z}}.$$

Extraction of coefficients is now easy:

$$[u^k]A(z,u) = \frac{t^k}{z^k}, \qquad [u^k]B(z,u) = \frac{t^{k+1}}{z^{k+2}}.$$

Furthermore

$$[z^nu^k]A(z,u) = [z^{n+k}]t^k$$

These coefficients are 0 unless $n+k = 3m$ for some $m \in \mathbb{N}$. Thus we will compute the coefficient of z^{3m-k} in $[u^k]A(z,u)$, and we write $x = z^3$ for convenience, as before.

$$\begin{aligned} [z^{3m-k}u^k]A(z,u) &= [z^{3m}]t^k = [x^m]t^k\\ &= [t^{m-k}]\frac{1-3t}{(1-t)^{2m+2}}\\ &= \binom{3m-k+1}{m-k} - 3\binom{3m-k}{m-k-1}. \end{aligned} \tag{1}$$

Likewise,

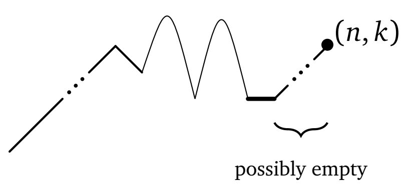
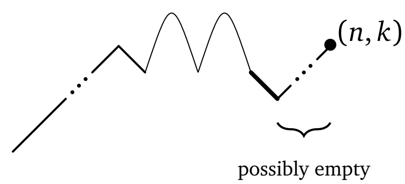
$$[z^nu^k]B(z,u) = [z^{n+k+2}]t^{k+1}.$$

Since this only makes sense if $n+k+2=3m$ for some $m\in\mathbb{N}$. Thus we read off the coefficient of z^{3m-k-2}:

$$[z^{3m-k-2}u^k]B(z,u)=[z^{3m}]t^{k+1}=[x^m]t^{k+1}=[t^{m-k-1}]\frac{1-3t}{(1-t)^{2m+2}}$$
$$=\binom{3m-k}{m-k-1}-3\binom{3m-k-1}{m-k-2}. \tag{2}$$

3.2. **Reverse S-Motzkin paths.** A *reverse S-Motzkin* path is an S-Motzkin path read from right to left. Alternatively, we might say that a reverse S-Motzkin path is a Motzkin path of length $3n$ with n of each type of step such that the first step from the step set $\{(1,0),(1,-1)\}$ is a $(1,-1)$ step. Furthermore, $(1,0)$ and $(1,-1)$ steps alternate.

Let $c_{n,k}$ denote a partial reverse S-Motzkin path of length n which ends at height k and the last step of the path in the step set $\{(1,0),(1,-1)\}$ is a $(1,0)$ step. Similarly, let $d_{n,k}$ denote a partial reverse S-Motzkin path of length n which ends at height k and the last step of the path in the step set $\{(1,0),(1,-1)\}$ is a $(1,-1)$ step.

The path represented by $c_{n,k}$	The path represented by $d_{n,k}$
	

It is easily seen that the following recurrence relations hold:

$$c_{n,k}=c_{n-1,k-1}+d_{n-1,k},$$
$$d_{n,k}=d_{n-1,k-1}+c_{n-1,k+1}.$$

These recurrences hold for $n\ge1$ and $k\ge0$; $c_{n-1,-1}$ and $d_{n-1,-1}$ must be interpreted as zero, and the initial values are $c_{0,0}=1$ and $d_{0,0}=0$.

Let

$$C(z,u)=\sum_{n\ge0}\sum_{k\ge0}c_{n,k}z^nu^k \qquad \text{and} \qquad D(z,u)=\sum_{n\ge0}\sum_{k\ge0}d_{n,k}z^nu^k,$$

and sum the recursion over n and k. This results in

$$C(z,u)-1=zuC(z,u)+zD(z,u),$$
$$D(z,u)=zuD(z,u)+\frac{z}{u}C(z,u)-\frac{z}{u}C(z,0).$$

Solving this system gives

$$C(z,u)=\frac{u-u^2z-C(z,0)z^2}{z^2u^3-2zu^2+u-z^2} \qquad \text{and} \qquad D(z,u)=\frac{C(z,0)uz^2-C(z,0)z+z}{z^2u^3-2zu^2+u-z^2}.$$

Note that the denominator is given by

$$z^2u^3 - 2zu^2 + u - z^2$$

whereas in the previous section (§3.1) the denominator was given by

$$z^2u^3 - u^2 + 2zu - z^2 = z^2(u - v_1)(u - v_2)(u - v_3);$$

the explicit forms of the roots are repeated for convenience:

$$v_1 = \frac{z}{t}, \qquad v_2 = -z\frac{t - 2 + \sqrt{4t - 3t^2}}{2(1-t)^2}, \qquad v_3 = -z\frac{t - 2 - \sqrt{4t - 3t^2}}{2(1-t)^2}.$$

The equation $z^2u^3 - u^2 + 2zu - z^2$ along with the substitution $u = 1/u$ and multiplication by $-u^3$ gives the denominator in the current case:

$$-u^3\Big(z^2\big(\tfrac{1}{u}\big)^3 - \big(\tfrac{1}{u}\big)^2 + 2z\big(\tfrac{1}{u}\big) - z^2\Big) = -z^2 + u - 2zu^2 + z^2u^3.$$

Therefore the roots of the polynomial $z^2u^3 - 2zu^2 + u - z^2$ are given by v_1^{-1}, v_2^{-1}, and v_3^{-1}, hence

$$z^2u^3 - 2zu^2 + u - z^2 = z^2\Big(u - \frac{1}{v_1}\Big)\Big(u - \frac{1}{v_2}\Big)\Big(u - \frac{1}{v_3}\Big).$$

Note that

$$\frac{1}{v_1} = \frac{t}{z}, \qquad \frac{1}{v_2} = \frac{-t + 2 + \sqrt{4t - 3t^2}}{2z}, \qquad \frac{1}{v_3} = \frac{-t + 2 - \sqrt{4t - 3t^2}}{2z}.$$

In the current right-to-left enumeration, $u - v_1^{-1}$ is the factor in the denominator that is also a factor of the numerator. Plugging in $u = v_1^{-1}$ into the numerator of $D(z,u)$ (the numerator of $C(z,u)$ could also be used) gives

$$C(z,0) = \frac{v_1}{v_1 - z}.$$

Using this value for $C(z,0)$, it follows that

$$\frac{C(z,0)uz^2 - C(z,0)z + z}{u - v_1^{-1}} = \frac{z^2v_1}{v_1 - z} = z^2C(z,0),$$

and thus

$$D(z,u) = \frac{C(z,0)}{\big(u - \frac{1}{v_2}\big)\big(u - \frac{1}{v_3}\big)}.$$

We can further write

$$D(z,u) = \frac{1}{(1-t)\big(u - \frac{1}{v_2}\big)\big(u - \frac{1}{v_3}\big)},$$

and representing this as a partial fraction gives

$$D(z,u) = \frac{t}{z(1-t)}\frac{v_3}{(1 - uv_3)(v_3 - v_2)} - \frac{t}{z(1-t)}\frac{v_2}{(1 - uv_2)(v_3 - v_2)}.$$

Therefore we can find the coefficients of $D(z,u)$:

$$[u^k]D(z,u)=\frac{t}{z(1-t)}\frac{v_3^{k+1}-v_2^{k+1}}{v_3-v_2}.$$

The identity [4, eq. (22)] will be useful in calculating $[z^nu^k]D(z,u)$ and $[z^nu^k]C(z,u)$, so note that

$$\begin{aligned}\frac{v_3^{k+1}-v_2^{k+1}}{v_3-v_2}&=\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i\binom{k-i}{i}(v_2+v_3)^{k-2i}(v_2v_3)^i\\&=\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^{i+k}\binom{k-i}{i}\frac{z^{k-2i}(t-2)^{k-2i}}{(t-1)^{2k-4i}}\frac{z^{2i}}{(1-t)^{2i}}\\&=z^k\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^{i+k}\binom{k-i}{i}\frac{(t-2)^{k-2i}}{(t-1)^{2k-2i}}.\end{aligned}$$

Further,

$$[u^k]D(z,u)=tz^{k-1}\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^{i+k-1}\binom{k-i}{i}\frac{(t-2)^{k-2i}}{(t-1)^{2k-2i+1}}.$$

Now set $n=3N+s-1$, $k=3K+s$ for $s\in\{0,1,2\}$. Then

$$\begin{aligned}[z^nu^k]D(z,u)&=[z^{3N+s-1}u^{3K+s}]D(z,u)\\&=[x^{N-K}]t\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^{i+k-1}\binom{k-i}{i}\frac{(t-2)^{k-2i}}{(t-1)^{2k-2i+1}}\\&=[x^{N-K}]t\sum_{i=0}^{\lfloor k/2\rfloor}\sum_{j=0}^{k-2i}(-1)^i\binom{k-i}{i}\binom{k-2i}{j}\frac{1}{(1-t)^{2k-2i-j+1}}\\&=\sum_{i=0}^{\lfloor k/2\rfloor}\sum_{j=0}^{k-2i}(-1)^i\binom{k-i}{i}\binom{k-2i}{j}\\&\qquad\times\left[\binom{n+k-2i-j+1}{(n-k+1)/3-1}-3\binom{n+k-2i-j}{(n-k+1)/3-2}\right].\end{aligned}\tag{3}$$

Similarly, for $C(z,u)$ we get

$$\frac{u-u^2z-C(z,0)z^2}{u-v_1^{-1}}=\frac{-(uv_1z-v_1+z)}{v_1},$$

and thus

$$C(z,u)=\frac{1-t-uz}{z^2\big(u-\frac{1}{v_2}\big)\big(u-\frac{1}{v_3}\big)}=\frac{(1-t-uz)}{(1-t)^2(1-v_2u)(1-v_3u)}.$$

Rewriting $C(z,u)$ using partial fractions gives

$$C(z,u) = \left[\frac{1}{1-t} - \frac{uz}{(1-t)^2}\right]\frac{1}{v_3-v_2}\left[\frac{v_3}{1-uv_3} - \frac{v_2}{1-uv_2}\right],$$

which allows for coefficient extraction:

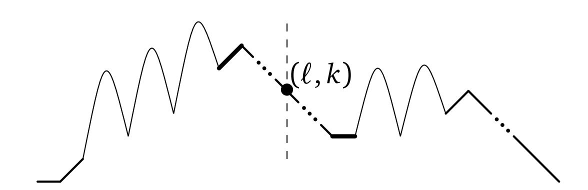
$$\begin{aligned}
[u^k]C(z,u) &= \frac{1}{1-t}\frac{v_3^{k+1}-v_2^{k+1}}{v_3-v_2} - \frac{z}{(1-t)^2}\frac{v_3^k-v_2^k}{v_3-v_2} \\
&= z^k\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^{i+k}\binom{k-i}{i}\frac{(t-2)^{k-2i}}{(t-1)^{2k-2i+1}} \\
&\quad - z^k\sum_{i=0}^{\lfloor (k-1)/2\rfloor}(-1)^{i+k-1}\binom{k-1-i}{i}\frac{(t-2)^{k-1-2i}}{(t-1)^{2k-2i}} \\
&= z^k\sum_{i=0}^{\lfloor k/2\rfloor}\sum_{j=0}^{k-2i}(-1)^{i-1}\binom{k-i}{i}\binom{k-2i}{j}\frac{1}{(1-t)^{2k-2i-j+1}} \\
&\quad - z^k\sum_{i=0}^{\lfloor (k-1)/2\rfloor}\sum_{j=0}^{k-1-i}(-1)^{i-1}\binom{k-i-1}{i}\binom{k-1-2i}{j}\frac{1}{(1-t)^{2k-2i-j}}.
\end{aligned}$$

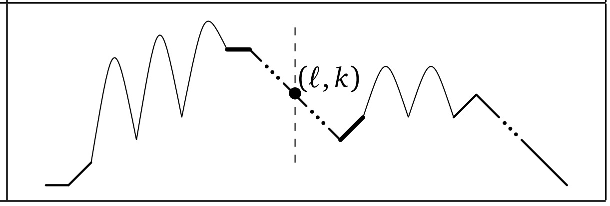
Furthermore, with $n = 3N+s$, $k = 3K+s$,

$$\begin{aligned}
[z^nu^k]C(z,u) &= [x^{N-K}]\sum_{i=0}^{\lfloor k/2\rfloor}\sum_{j=0}^{k-2i}(-1)^i\binom{k-i}{i}\binom{k-2i}{j}\frac{1}{(1-t)^{2k-2i-j+1}} \\
&\quad - [x^{N-K}]\sum_{i=0}^{\lfloor (k-1)/2\rfloor}\sum_{j=0}^{k-1-i}(-1)^i\binom{k-1-i}{i}\binom{k-1-2i}{j}\frac{1}{(1-t)^{2k-2i-j}} \\
&= \sum_{i=0}^{\lfloor k/2\rfloor}\sum_{j=0}^{k-2i}(-1)^i\binom{k-i}{i}\binom{k-2i}{j} \qquad (4)\\
&\quad \times\left[\binom{n+k-2i-j+1}{(n-k)/3} - 3\binom{n+k-2i-j}{(n-k)/3-1}\right] \\
&\quad - \sum_{i=0}^{\lfloor (k-1)/2\rfloor}\sum_{j=0}^{k-1-i}(-1)^i\binom{k-1-i}{i}\binom{k-1-2i}{j} \qquad (5)\\
&\quad \times\left[\binom{n+k-2i-j}{(n-k)/3} - 3\binom{n+k-2i-j-1}{(n-k)/3-1}\right]. \qquad (6)
\end{aligned}$$

3.3. **The area of S-Motzkin paths.** Consider an arbitrary S-Motzkin path of length $3n$. This path can be decomposed at any height k that the path attains into either an (1) $a_{\ell,k}$ path followed by a $c_{3n-\ell,k}$ path or (2) a $b_{\ell,k}$ path followed by a $d_{3n-\ell,k}$ path.

The subpaths $a_{\ell,k}$ and $c_{3n-\ell,k}$	The subpaths $b_{\ell,k}$ and $d_{3n-\ell,k}$





We see that the generating function for the total area (all the areas, summed over all S-Motzkin paths of the same length) is given by

$$\sum_{k\ge0} k\Big([u^k]A(z,u)\cdot[u^k]C(z,u)+[u^k]B(z,u)\cdot[u^k]D(z,u)\Big).$$

Fortunately, we have explicit forms for $[u^k]A(z,u)$, $[u^k]B(z,u)$, $[u^k]C(z,u)$, $[u^k]D(z,u)$, and we compute the sum, using the forms of v_2 and v_3 involving t (using a computer, of course):

$$\begin{aligned}
&\sum_{k\ge0} k\Big([u^k]A(z,u)\cdot[u^k]C(z,u)+[u^k]B(z,u)\cdot[u^k]D(z,u)\Big)\\
&=\sum_{k\ge0} k\Bigg(\Big(\frac{1}{zv_2^{k+1}v_3^{k+1}}-\frac{1}{zv_2^{k+2}v_3^{k+2}}\Big)\cdot\Big((v_3^{k+1}-v_2^{k+1})\Big(\frac{v_2v_3-v_2^2v_3^2z}{z^2(v_3-v_2)}\Big)+(v_2^k-v_3^k)\Big(\frac{v_2v_3}{z(v_3-v_2)}\Big)\Big)\\
&\qquad+\Big(\frac{1}{zv_2^{k+1}v_3^{k+1}}\Big)\cdot(v_3^{k+1}-v_2^{k+1})\Big(\frac{v_2v_3}{(v_3-v_2)(1-zv_2v_3)}\Big)\Bigg)=\frac{t}{(1-t)^2(1-3t)^2}.
\end{aligned}$$

Using Cauchy's integral formula we find that the area of S-Motzkin paths of length $3n$ is given by

$$\begin{aligned}
[x^n]\frac{t}{(1-t)^2(1-3t)^2}&=\frac{1}{2\pi i}\oint\frac{dx}{x^{n+1}}\frac{t}{(1-t)^2(1-3t)^2}\\
&=\frac{1}{2\pi i}\oint\frac{dt}{t^n(1-t)^{2n+3}}\frac{1}{(1-3t)}\\
&=[t^{n-1}]\frac{1}{(1-t)^{2n+3}}\frac{1}{(1-3t)}\\
&=\sum_{k=0}^{n-1}3^k[t^{n-1-k}]\frac{1}{(1-t)^{2n+3}}\\
&=\sum_{k=0}^{n-1}3^k\binom{3n+1-k}{n-1-k}. \qquad (7)
\end{aligned}$$

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