IDENTITIES OF GONZALEZ PROVED BY A POWER SERIES APPROACH

HELMUT PRODINGER

Abstract. Gonzalez provided an elementary approach to prove a class of combinatorial identities. We offer as an alternative a method that is sometimes called binomial transform.

1. Introduction

Gonzalez [1] presented an elementary method to evaluate sums of the type

$$b_n := \sum_{k=0}^{n} a_k \binom{n}{k},$$

for various sequences a_n. His approach is based on the recursion

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

for Pascal's triangle.

The reader is invited to compare this with a very elegant and simple method that uses manipulations on power series ("generating functions"). This is well within the reach of beginning students and requires no special skills. Only the two series expansions

$$\frac{1}{(1-t)^{m+1}} = \sum_{n\ge0} t^n \binom{n+m}{n}, \qquad \log\frac{1}{1-t} = \sum_{n\ge1}\frac{t^n}{n}$$

need to be known. A general reference that I would like to mention is the classic book [2].

Set

$$f(t) := \sum_{n\ge0} a_n t^n \quad \text{and} \quad g(t) := \sum_{n\ge0} b_n t^n.$$

Then

$$g(t) = \frac{1}{1-t} f\Big(\frac{t}{1-t}\Big).$$

The justification is easy:

$$\begin{aligned} g(t) &= \sum_{n\ge0}\sum_{k=0}^{n} a_k \binom{n}{k} t^n = \sum_{k\ge0} a_k \sum_{n\ge k}\binom{n}{k} t^n = \sum_{k\ge0} a_k \frac{t^k}{(1-t)^{k+1}} \\ &= \frac{1}{1-t}\sum_{k\ge0} a_k \Big(\frac{t}{1-t}\Big)^k = \frac{1}{1-t} f\Big(\frac{t}{1-t}\Big). \end{aligned}$$

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Based on this simple correspondence (which is sometimes called binomial transform) we will demonstrate how to deal with the examples presented in [1].

We will use the functions $f(t)$ and $g(t)$ exactly in the same way as described. An important notation is also $[t^n]F(t)$, which is the coefficient of t^n in the power series $F(t)$.

2. Examples

Identity 1.

$$\sum_{k=0}^{n}\binom{n}{k}=2^n.$$

Here, $a_n=1$, whence $f(t)=\sum_{n\ge0}t^n=\frac{1}{1-t}$, and thus

$$g(t)=\frac{1}{1-t}\frac{1}{1-\frac{t}{1-t}}=\frac{1}{1-2t},$$

so that

$$b_n=[t^n]g(t)=2^n.$$

Identity 2.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

for $n\ge1$. Here, $a_n=(-1)^n$, whence $f(t)=\frac{1}{1+t}$, and thus

$$g(t)=\frac{1}{1-t}\frac{1}{1+\frac{t}{1-t}}=1,$$

which is the result.

Identity 3.

$$\sum_{k=0}^{n}k\binom{n}{k}=n2^{n-1}$$

for $n\ge1$. Here, $a_n=n$, whence $f(t)=\sum_{n\ge0}nt^n=\frac{t}{(1-t)^2}$, and thus

$$g(t)=\frac{t}{(1-2t)^2}.$$

Consequently

$$b_n=[t^n]g(t)=[t^{n-1}]\frac{1}{(1-2t)^2}=n2^{n-1}.$$

Identity 4. Let

$$S_{n,r}:=\sum_{k=0}^{n}k^r\binom{n}{k}.$$

Then

$$f_r(t)=\sum_{n\ge0}n^rt^n=\Big(t\frac{d}{dt}\Big)^r\frac{1}{1-t}=t\frac{d}{dt}f_{r-1}(t)$$

and

$$\begin{aligned} g_r(t) &= \frac{1}{1-t} f_r\Big(\frac{t}{1-t}\Big) = \frac{1}{1-t}\frac{t}{1-t}\frac{d}{dx} f_{r-1}(x)\Big|_{x=\frac{t}{1-t}} \\ &= \frac{t}{(1-t)^2}\frac{dt}{dx}\cdot\frac{d}{dt} f_{r-1}\Big(\frac{t}{1-t}\Big) = t\frac{d}{dt}(1-t)g_{r-1}(t). \end{aligned}$$

Therefore

$$\begin{aligned} S_{n,r} &= [t^n]g_r(t) = [t^{n-1}]\Big(g'_{r-1}(t) - tg'_{r-1}(t) - g_{r-1}(t)\Big) \\ &= nS_{n,r-1} - (n-1)S_{n-1,r-1} - S_{n-1,r-1} = n(S_{n,r-1} - S_{n-1,r-1}). \end{aligned}$$

Identity 5. Consider

$$\sum_{k=0}^{n} \binom{m}{r-k}\binom{n}{k}.$$

Here, $a_n = \binom{m}{r-n}$ and

$$f(t) = \sum_{n\ge 0}\binom{m}{r-n} t^n = \sum_{n=0}^{r}\binom{m}{n} t^{r-n} = t^r\Big(1+\frac{1}{t}\Big)^m = t^{r-m}(1+t)^m.$$

Consequently

$$g(t) = \frac{1}{1-t}\frac{t^{r-m}}{(1-t)^{r-m}}\Big(\frac{1}{1-t}\Big)^m = \frac{t^{r-m}}{(1-t)^{r+1}}$$

and

$$b_n = [t^n]g(t) = [t^{n-r+m}]\frac{1}{(1-t)^{r+1}} = \binom{n+m}{n-r+m} = \binom{n+m}{r}.$$

Identity 6. Consider

$$\sum_{k=0}^{n} k\binom{m}{r-k}\binom{n}{k}.$$

Here, $a_n = n\binom{m}{r-n}$ and

$$f(t) = \sum_{n\ge 0} n\binom{m}{r-n} t^n = t\frac{d}{dt} t^{r-m}(1+t)^m = rt^{r+1-m}(1+t)^{m-1} + (r-m)t^{r-m}(1+t)^{m-1}.$$

Therefore

$$g(t) = \frac{rt^{r+1-m}}{(1-t)^{r+1}} + \frac{(r-m)t^{r-m}}{(1-t)^r}$$

and

$$\begin{aligned} b_n &= [t^n]g(t) = r\binom{n+m-1}{n-r-1+m} + (r-m)\binom{n+m-1}{n-r+m} \\ &= r\binom{n+m-1}{r} + (r-m)\binom{n+m-1}{r-1} = n\binom{n+m-1}{r-1}. \end{aligned}$$

Identity 7. Consider

$$\sum_{k=0}^{n} F_k\binom{n}{k},$$

with a Fibonacci number. Here, $a_n = F_n$ and

$$f(t) = \sum_{n\ge 0} t^n F_n = \frac{t}{1-t-t^2} = \frac{1}{\sqrt{5}(1-\alpha t)} - \frac{1}{\sqrt{5}(1-\beta t)},$$

with

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

Consequently

$$g(t) = \frac{t}{t^2-3t+1} = \frac{1}{\sqrt{5}(1-\alpha^2 t)} - \frac{1}{\sqrt{5}(1-\beta^2 t)},$$

and thus

$$b_n = [t^n]g(t) = F_{2n}.$$

Identity 8. Consider

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k},$$

then

$$f(t) = \sum_{n\ge 1} t^n \frac{(-1)^{n+1}}{n} = \log(1+t),$$

and

$$g(t) = \frac{1}{1-t}\log\Big(1+\frac{t}{1-t}\Big) = \frac{1}{1-t}\log\frac{1}{1-t},$$

and

$$b_n = [t^n]\frac{1}{1-t}\log\frac{1}{1-t} = \sum_{k=1}^{n}[t^k]\log\frac{1}{1-t} = \sum_{k=1}^{n}\frac{1}{k} = H_n,$$

a harmonic number.

Identity 9. Consider

$$\sum_{k=1}^{n} H_k\binom{n}{k},$$

then

$$f(t) = \frac{1}{1-t}\log\frac{1}{1-t}$$

and

$$g(t) = \frac{1}{1-2t}\log\frac{1-t}{1-2t}.$$

Therefore

$$b_n = [t^n]g(t) = 2^n H_n - \sum_{k=1}^{n}\frac{1}{k}2^{n-k} = 2^n\Big(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\Big).$$

The number of examples could of course be endless, but we decided to stop here.

References

[1] L. Gonzalez, *A new approach for proving or generating combinatorial identities*, International Journal of Mathematical Education in Science and Technology 41 (2010), 359–372.
[2] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics (Second Edition)*. Addison Wesley, 1994.

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