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Note

On the Number of Combinations without a Fixed Distance

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An explicit formula is derived for the number of k-element subsets A of $\{1, 2,..., n\}$ such that no two elements whose difference is q are in A.

1. INTRODUCTION

In a recent paper Konvalina [2] has determined $f(n, k)$, the number of k-element subsets A of $\{1, 2,..., n\}$ such that $i, i+2 \in A$ is impossible ("no unit separation"). His answer was obtained by guessing the correct formula (1) and proving it afterward:

$$\begin{aligned} f(n,k) &= \sum_{0 \leqslant i \leqslant k/2} \binom{n+1-k-2i}{k-2i} & &\text{if } 0 \leqslant k \leqslant \frac{n}{2}+1, \\ &= 0 & &\text{otherwise.} \end{aligned} \tag{1}$$

The similar problem where $i, i+1 \in A$ is prohibited is quite well known and easily solved, too: If $A = \{i_1, i_2,..., i_k\}$ is a correct subset, $\{i_1, i_2 - 1,..., i_k - (k-1)\}$ is a subset of $\{1,..., n-(k-1)\}$, hence there are $\binom{n+1-k}{k}$ such subsets.

Our basic idea to attack Konvalina's problem is simply to dissect $\{1, 2,..., n\}$ into $\{1, 3,...\}$ and $\{2, 4,...\}$. So a Konvalina subset consists of odd and of even elements; two consecutive odd (even) integers are prohibited. This means that we can reduce the problem to the classical one by means of the convolution of the two parts (odd, resp. even).

In quite the same style we can solve the general exercise where $i, i+q$ is prohibited. Let us denote the corresponding number of subsets by $f_q(n, k)$.

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Before pointing out our main result let us make some conventions: We write $[x]$ for the largest integer less than or equal to x; if $\Phi(z)$ is a power series, $[z^k]\,\Phi$ denotes the coefficient of z^k in the power series Φ.

We will use a complex variable theory method which was used by de Bruijn *et al.* in a context of another problem [1]. We need Cauchy's integral formula, viz.

$$[z^k]\,\Phi(z) = \frac{1}{2\pi i}\int \frac{dz}{z^{k+1}}\,\Phi(z), \tag{2}$$

where the path of integration is a contour encircling the origin inside the domain of analyticity.

THEOREM. *Let* $m = [n/q]$ *and* $d = n - q[n/q]$. *Then*

$$f_q(n,k) = \sum (-1)^{\lambda_1 + m\lambda_3 + (m+1)\lambda_4} \times \binom{q-2+\lambda_1}{\lambda_1}\binom{n+q-2k+\lambda_2}{\lambda_2}\binom{d}{\lambda_3}\binom{q-d}{\lambda_4}, \tag{3}$$

where the sum is about all λ_i such that $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, $0 \leqslant \lambda_3 \leqslant d$, $0 \leqslant \lambda_4 \leqslant q-d$: $\lambda_1 + \lambda_2 + \lambda_3(m+3) + \lambda_4(m+2) = k$ provided that $0 \leqslant k \leqslant \frac{1}{2}(n+q)$; otherwise $f_q(n,k) = 0$.

2. RESULTS

The generating function

$$\Phi_n(x) := \sum_{k \geqslant 0} \binom{n+1-k}{k}(-x)^k \tag{4}$$

of the numbers of k-subsets of $\{1, 2,..., n\}$ without adjacent elements admits the following closed form representation (cf. Riordan [3, p. 76])

$$\Phi_n(x) = \frac{1}{\alpha}\left[\left(\frac{1+\alpha}{2}\right)^{n+2} - \left(\frac{1-\alpha}{2}\right)^{n+2}\right], \quad \text{with} \quad \alpha = \sqrt{1-4x}. \tag{5}$$

If we substitute $x = u/(1+u)^2$ we obtain

$$\Phi_n(x) = \frac{1+u}{1-u}\cdot\frac{1-u^{n+2}}{(1+u)^{n+2}}. \tag{6}$$

To compute $f_q(n,k)$, we write $n = qm + d$, with $0 \leqslant d < q$ and dissect $\{1,..., n\}$ according to the residue classes mod q. There are d classes with

$m+1$ and $q-d$ classes with m elements. According to our reduction technique this means in terms of generating functions

$$f_q(n, k) = (-1)^k [x^k]\, \Phi_{m+1}^d(x)\, \Phi_m^{q-d}(x). \tag{7}$$

The range of k (obtained by this dissection into residue classes is $0 \leqslant k \leqslant d/2 + q(m+1)/2 = \frac{1}{2}(n+q)$. To compute $f_q(n, k)$ we use (6) and Cauchy's formula,

$$\begin{aligned} f_q(n, k) &= (-1)^k \frac{1}{2\pi i} \int \frac{dx}{x^{k+1}} \left(\frac{1+u}{1-u}\right)^q \frac{(1-u^{m+3})^d}{(1+u)^{(m+3)d}} \frac{(1-u^{m+2})^{q-d}}{(1+u)^{(m+2)(q-d)}} \\ &= (-1)^k \frac{1}{2\pi i} \int \frac{du}{u^{k+1}} \cdot \frac{(1-u)(1+u)^{2k+2}}{(1+u)^3} \\ &\quad \cdot \frac{(1-u^{m+3})^d\, (1-u^{m+2})^{q-d}}{(1-u)^q\, (1+u)^{d+q(m+1)}} \\ &= (-1)^k [u^k] \frac{(1-u^{m+3})^d \cdot (1-u^{m+2})^{q-d}}{(1-u)^{q-1}(1+u)^{n+q-2k+1}}. \end{aligned} \tag{8}$$

It is a matter of routine to expand the rational function in (8) according to the binomial theorem, yielding the proof of the Theorem.

Remark. In Konvalina's case $q = 2$ the result simplifies very much, since nonzero contributions in (3) are only obtained for $\lambda_3 = \lambda_4 = 0$, so that we have

$$\begin{aligned} f_2(n, k) &= \sum_{\substack{\lambda_1 \geqslant 0; \lambda_2 \geqslant 0 \\ \lambda_1 + \lambda_2 = k}} (-1)^{\lambda_1} \binom{n+2-2k+\lambda_2}{\lambda_2} \\ &= \sum_{0 \leqslant \lambda \leqslant k} (-1)^{\lambda} \binom{n+2-k-\lambda}{k-\lambda}. \end{aligned} \tag{9}$$

This is not Konvalina's solution (1), but it is equivalent. We group together two adjacent terms in (9): $\lambda = 0, 1$; $\lambda = 2, 3$ and so on. Writing $\lambda = 2j, 2j+1$ we have

$$\binom{n+2-k-2j}{k-2j} - \binom{n+1-k-2j}{k-1-2j} = \binom{n+1-k-2j}{k-2j},$$

and the equivalence of (1) and (9) is clear for k odd (then in (9) we have an even number of summands). If k is even, there is one extra summand in (9) and (1), respectively; both terms are 1, so that we have found again Konvalina's result.

References

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