A GENERALIZATION OF A FILBERT MATRIX WITH 3 ADDITIONAL PARAMETERS

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Abstract. A generalized Filbert matrix from [1] is further generalized, introducing 3 additional parameters. Explicit formulæ are derived for the LU-decomposition, their inverses, and the Cholesky factorization. The approach is to use q-analysis; the necessary identities are clarified on 2 examples and can otherwise be left to automatic proofs (q-Zeilberger algorithm).

1. Introduction

In [1], a *generalized Filbert matrix* with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter, was studied (F_n is the n-th Fibonacci number). The size of the matrix does not really matter, and we can think about an infinite matrix $\mathcal{F}$ and restrict it whenever necessary to the first n rows resp. columns and write $\mathcal{F}_n$. See [1] for historic remarks and pointers to the earlier literature.

Throughout, we will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

In the previous paper, explicit formulæ where given for:

- The LU-decomposition $\mathcal{F} = L \cdot U$.
- The inverse matrices L^{-1}, U^{-1}.
- The inverse of the matrix $\mathcal{F}_n$.
- The Cholesky decomposition $\mathcal{F} = \mathcal{C} \cdot \mathcal{C}^T$ with a lower triangular matrix $\mathcal{C}$.

All the identities hold for general q, and results about Fibonacci numbers come out as corollaries for the special choice of q. Henceforth, we don't mention Fibonacci numbers any further.

In this paper, we generalize all this by introducing *three* extra parameters: x, y, and λ:

$$\mathcal{F}_{i,j} := \frac{x^i y^j}{F_{\lambda(i+j)+r}}.$$

The previous instances come out as the special cases $x = y = \lambda = 1$. (For the Cholesky case, we have to naturally assume that $x = y$.)

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The main effort in obtaining these results is to guess the correct formulæ, which is doable with some experience, patience, and a computer. The proofs are then *routine*, and in our previous paper, we only mentioned that the q-Zeilberger algorithm can do it. Here, we want to be a bit more explicit, for the convenience of the reader.

Throughout this paper we will use the following notations: $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Sometimes, when the parameter q is understood, instead of $(x;q)_n$, only $(x)_n$ is written.

2. Results

Theorem 1. *The LU decomposition is given by*

$$L_{n,d} = \mathbf{i}^{\lambda(d-n)} q^{\frac{\lambda(n-d)}{2}} \frac{(q^{\lambda+r};q^\lambda)_{2d}(q^{\lambda+r};q^\lambda)_n}{(q^{\lambda+r};q^\lambda)_{n+d}(q^{\lambda+r};q^\lambda)_d} \frac{(q^\lambda;q^\lambda)_{n-1}}{(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{d-1}} y^{n-d}$$

and

$$U_{d,n} = \frac{(q^\lambda;q^\lambda)_{d-1}(q^{\lambda+r};q^\lambda)_d(q^{\lambda+r};q^\lambda)_n(q^\lambda;q^\lambda)_n(1-q)}{(q^{\lambda+r};q^\lambda)_{2d-1}(q^{\lambda+r},q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}(1-q^{\lambda n})} x^n y^d \\ \times q^{\frac{\lambda(n-d)}{2}+\lambda d^2+rd-\frac{r+1}{2}} \mathbf{i}^{\lambda(n+d)+r+1}(-1)^{\lambda(n+d)+r}$$

for $n \ge d$; for $n < d$, these numbers are 0.

Theorem 2. *The inverse matrices are given by*

$$L^{-1}_{n,d} = q^{\lambda\frac{(n-d)^2}{2}} \mathbf{i}^{\lambda(n+d)}(-1)^{(\lambda+1)n+d} \frac{(q^{\lambda+r};q^\lambda)_{n+d-1}(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n}{(q^{\lambda+r};q^\lambda)_d(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{d-1}(q^{\lambda+r};q^\lambda)_{2n-1}} y^{n-d}$$

and

$$U^{-1}_{d,n} = \frac{(q^{\lambda+r};q^\lambda)_{2n}(q^{\lambda+r};q^\lambda)_{n+d-1}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n(q^{\lambda+r};q^\lambda)_d} \frac{1}{1-q} x^{-d} y^{-n} \\ \times q^{-\frac{\lambda n^2}{2}+\frac{\lambda d^2}{2}+\frac{r+1}{2}-\lambda dn-rn} \mathbf{i}^{\lambda(n+d)+r-1}(-1)^{n+d}$$

for $n \ge d$; for $n < d$, these numbers are 0.

Theorem 3. *The inverse matrix:*

$$(\mathcal{F}_n)^{-1}_{i,j} = q^{\frac{\lambda i^2+\lambda j^2+r+1}{2}-\lambda(i+j)n-rn} \mathbf{i}^{\lambda i+\lambda j+r+1}(-1)^{i+j+1} \frac{1}{(1-q^{r+\lambda i+\lambda j})(1-q)} x^{-i} y^{-j} \\ \times \frac{(q^{\lambda+r};q^\lambda)_{n+i}(q^{\lambda+r};q^\lambda)_{n+j}}{(q^\lambda;q^\lambda)_{i-1}(q^\lambda;q^\lambda)_{n-i}(q^\lambda;q^\lambda)_{j-1}(q^\lambda;q^\lambda)_{n-j}(q^{\lambda+r};q^\lambda)_i(q^{\lambda+r};q^\lambda)_j}.$$

Theorem 4. *The Cholesky decomposition is for $x = y$ given by*

$$\mathcal{C}_{n,d} = (-1)^n \mathbf{i}^{\lambda n+r+\frac{r+1}{2}} q^{\frac{\lambda n}{2}-\frac{r+1}{4}+\frac{\lambda d(d-1)}{2}+\frac{rd}{2}} \sqrt{(1-q^{2\lambda d+r})(1-q)} \frac{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n}{(q^\lambda;q^\lambda)_{n-d}(q^{\lambda+r};q^\lambda)_{n+d}} x^n.$$

This holds for $n \ge d$; otherwise, these numbers are 0.

3. How the prove the formulæ

Let us consider one typical case:

$$\sum_d L_{m,d} L^{-1}_{d,n} = \sum_d \mathbf{i}^{\lambda(d-m)} q^{\frac{\lambda(m-d)}{2}} \frac{(q^{\lambda+r};q^\lambda)_{2d}(q^{\lambda+r};q^\lambda)_m}{(q^{\lambda+r};q^\lambda)_{m+d}(q^{\lambda+r};q^\lambda)_d} \frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{m-d}(q^\lambda;q^\lambda)_{d-1}} y^{m-d}$$

$$\times q^{\lambda\frac{(n-d)^2}{2}} \mathbf{i}^{\lambda(n+d)} (-1)^{(\lambda+1)d+n} \frac{(q^{\lambda+r};q^\lambda)_{n+d-1}(q^\lambda;q^\lambda)_{d-1}(q^{\lambda+r};q^\lambda)_d}{(q^{\lambda+r};q^\lambda)_n(q^\lambda;q^\lambda)_{d-n}(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_{2d-1}} y^{d-n}$$

$$= q^{\frac{\lambda m}{2}+\frac{\lambda n^2}{2}} y^{m-n} (-1)^n \mathbf{i}^{\lambda(n-m)} \frac{(q^{\lambda+r};q^\lambda)_m(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n}$$

$$\times \sum_d q^{\frac{\lambda d^2}{2}-\frac{\lambda d}{2}-\lambda dn} \frac{(-1)^d}{(q^{\lambda+r};q^\lambda)_{m+d}} \frac{1-q^{r+2d\lambda}}{(q^\lambda;q^\lambda)_{m-d}} \frac{(q^{\lambda+r};q^\lambda)_{n+d-1}}{(q^\lambda;q^\lambda)_{d-n}}.$$

It is clear that, since the matrices are of lower triangular type, we have to prove that the expression is 0 for $m > n$. (For $m = n$, the entry is 1, which is not hard to see, since it is only one term, not a sum.)

It is beneficial, for the evaluation of the sum, to switch to more convenient letters. First, set $Q = q^\lambda$ and $a = q^r$:

$$\mathsf{SUM} = \sum_d Q^{\binom{d}{2}-dn} \frac{(-1)^d}{(aQ;Q)_{m+d}} \frac{1-aQ^{2d}}{(Q;Q)_{m-d}} \frac{(aQ;Q)_{n+d-1}}{(Q;Q)_{d-n}}.$$

Now, we write q for Q and k for d:

$$\mathsf{SUM} = \sum_{k=n}^{m} q^{\binom{k}{2}-kn} (-1)^k \frac{(1-aq^{2k})(aq)_{n+k-1}}{(aq)_{m+k}(q)_{m-k}(q)_{k-n}}.$$

Changing k to $k+n$ (and introducing an irrelevant factor):

$$\mathsf{SUM} = \sum_{k=0}^{m-n} q^{\binom{k}{2}} (-1)^k \frac{(1-aq^{2k+2n})(aq)_{2n+k-1}}{(aq)_{m+n+k}(q)_{m-n-k}(q)_k}.$$

Changing m to $m+n$ (and introducing an irrelevant factor):

$$\mathsf{SUM} = \sum_{k=0}^{m} q^{\binom{k}{2}} (-1)^k \frac{1-aq^{2k+2n}}{(aq^{2n+k})_{m+1}} \begin{bmatrix} m \\ k \end{bmatrix}.$$

Eventually, we replace aq^{2n} by b:

$$\mathsf{SUM} = \sum_{k=0}^{m} q^{\binom{k}{2}} (-1)^k \frac{1-bq^{2k}}{(bq^k)_{m+1}} \begin{bmatrix} m \\ k \end{bmatrix}.$$

We must show that this is 0 whenever $m>0$. Now set

$$F(m,k)=q^{\binom{k}{2}}(-1)^k\frac{1-bq^{2k}}{(bq^k)_{m+1}}\begin{bmatrix}m\\k\end{bmatrix},$$
$$G(m,k)=q^{\binom{k+1}{2}}(-1)^k\frac{1}{(bq^{k+1})_m}\begin{bmatrix}m-1\\k\end{bmatrix},$$

then it is trivial to check that

$$F(m,k)=G(m,k)-G(m,k-1);$$

this is the representation that the q-Zeilberger algorithm provides. Summing this over all k gives the result 0, as the righthand side telescopes. Alternatively, one can notice that the sum of interest is given by $G(m,m)$, which evaluates to 0.

Let us consider another case:

$$\begin{aligned}\sum_{m\le d\le n}U_{m,d}U^{-1}_{d,n}&=\sum_{m\le d\le n}\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda+r};q^\lambda)_m(q^{\lambda+r};q^\lambda)_d(q^\lambda;q^\lambda)_d(1-q)}{(q^{\lambda+r};q^\lambda)_{2m-1}(q^{\lambda+r},q^\lambda)_{d+m}(q^\lambda;q^\lambda)_{d-m}(1-q^{\lambda d})}x^dy^m\\
&\quad\times q^{\frac{\lambda(d-m)}{2}+\lambda m^2+rm-\frac{r+1}{2}}\mathbf{i}^{\lambda(d+m)+r+1}(-1)^{\lambda(d+m)+r}\\
&\quad\times\frac{(q^{\lambda+r};q^\lambda)_{2n}(q^{\lambda+r};q^\lambda)_{n+d-1}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n(q^{\lambda+r};q^\lambda)_d}\frac{1}{1-q}x^{-d}y^{-n}\\
&\quad\times q^{-\frac{\lambda n^2}{2}+\frac{\lambda d^2}{2}+\frac{r+1}{2}-\lambda dn-rn}\mathbf{i}^{\lambda(n+d)+r-1}(-1)^{n+d}\\
&=\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda+r};q^\lambda)_m(q^{\lambda+r};q^\lambda)_{2n}}{(q^{\lambda+r};q^\lambda)_{2m-1}(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n}y^{m-n}q^{-\frac{\lambda m}{2}+\lambda m^2+rm-\frac{\lambda n^2}{2}-rn}\mathbf{i}^{\lambda(m+n)}(-1)^{\lambda m+n}\\
&\quad\times\sum_{m\le d\le n}\frac{(-1)^dq^{\frac{\lambda d^2}{2}+\frac{\lambda d}{2}-\lambda dn}(q^{\lambda+r};q^\lambda)_{n+d-1}}{(q^{\lambda+r};q^\lambda)_{d+m}(q^\lambda;q^\lambda)_{d-m}(q^\lambda;q^\lambda)_{n-d}}.\end{aligned}$$

For the sum, let us write $Q=q^\lambda$ and $a=q^r$:

$$\sum_{m\le d\le n}\frac{(-1)^dq^{\frac{d^2}{2}+\frac{d}{2}-dn}(aQ;Q)_{n+d-1}}{(aQ;Q)_{d+m}(Q;Q)_{d-m}(Q;Q)_{n-d}}.$$

We must show that it is zero for $m<n$. Let us replace d by $d+m$, ignore irrelevant factors and write q again for Q for convenience:

$$\sum_{0\le d\le n-m}\frac{(-1)^dq^{\frac{d^2}{2}+\frac{d}{2}+dm-dn}(aq)_{n+d+m-1}}{(aq)_{d+2m}(q)_d(q)_{n-d-m}}.$$

Now replace n by $n+m$:

$$\sum_{0\le d\le n}\frac{(-1)^dq^{\frac{d^2}{2}+\frac{d}{2}-dn}(aq)_{n+d+2m-1}}{(aq)_{d+2m}(q)_d(q)_{n-d}}.$$

Ignore irrelevant factors:

$$\sum_{0\le d\le n}(-1)^d q^{\frac{d^2}{2}+\frac{d}{2}-dn}(aq^{1+d+2m})_{n-1}\begin{bmatrix}n\\d\end{bmatrix}.$$

Now write $b=aq^{1+2m}$ and the more traditional k instead of d:

$$\sum_{k=0}^{n}(-1)^k q^{\frac{k^2}{2}+\frac{k}{2}-kn}(bq^k)_{n-1}\begin{bmatrix}n\\k\end{bmatrix}.$$

As before, call the term in the sum $F(n,k)$. Then

$$F(n,k)=G(n,k)-G(n,k-1),$$

with

$$G(n,k)=(-1)^k q^{\frac{k^2}{2}+\frac{k}{2}-kn}(bq^k)_n\frac{1}{1-bq^{n-1}}\begin{bmatrix}n-1\\k\end{bmatrix}.$$

Again, upon summing on k, we get 0, because of the telescoping property.

Other proofs follow the same pattern.

References

[1] E. Kilic and H. Prodinger, A generalized Filbert matrix, Fibonacci Quart. 48 (2010), 29–33.

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