A GENERALIZATION OF A CONJECTURE OF MELHAM

EMRAH KILIC[1], ILKER AKKUS[2], AND HELMUT PRODINGER[3]

Abstract. A generalization of one of Melham's conjectures is presented. After writing it in terms of Gaussian q–binomial coefficients, a solution is found using the elementary technique of partial fraction decomposition.

1. Introduction

The *Fibonomial coefficient* is, for $n \geq m \geq 1$, defined by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} := \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_{n-m})\,(F_1 F_2 \dots F_m)}$$

with $\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 1$ where F_n is the nth Fibonacci number .

For a detailed discussion of the Fibonomial coefficients, we refer to the list of references in [2].

The Gaussian q–binomial coefficient $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is defined, for all real n and integers m with $m \geq 0$, by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}$$

and as zero otherwise, where

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}).$$

Thus, $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is a rational function of the parameter q. For more details, see [1].

Let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then the well known Binet forms give

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n,$$

and thus

$$F_n = \alpha^{n-1}\frac{1-q^n}{1-q}, \qquad L_n = \alpha^n(1+q^n),$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\mathbf{i} = \alpha\sqrt{q}$, where F_n is the nth Fibonacci number and L_n is the nth Lucas number. All the identities we are going to derive hold for

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general q, and results about Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

The link between the Fibonomial and Gaussian q–binomial coefficients is

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \alpha^{m(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

Melham [3] derived some families of identities between sums of powers of the Fibonacci and Lucas numbers. He also conjectured a complex identity by using a "falling" Fibonacci factorial $(F_n)_{(m)}$, which begins at F_n for $n \neq 0$, and is the product of m Fibonacci numbers excluding F_0. His conjecture is in two parts: Let $m, n, k \in \mathbb{Z}$ with $m \geq 1$. Then:

(a)

$$\sum_{j=0}^{m-1} \frac{F_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m-1)}{2}} \frac{F_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = F_{(m+1)(n+\frac{m}{2})}.$$

(b) The Lucas counterpart of (a),

$$\sum_{j=0}^{m-1} \frac{L_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m-1)}{2}} \frac{L_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = \begin{cases} 5^{\frac{m+1}{2}} F_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} L_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is even.} \end{cases}$$

In [2], we rearranged the conjectures (a) and (b) by using Fibonomials and then gave a solution of the conjecture by translating it into a q–expression: we were left with the evaluation of a certain sum. This was achieved using contour integration.

The present paper is organized as follows: (i) We generalize the conjecture of Melham by using indices in arithmetic progression for both, the Fibonacci and the Lucas instance. (ii) Then we give a solution for this general formula by a partial fraction decomposition method that is even simpler than the contour integration given in [2] (although it is essentially equivalent).

2. A generalization of Melham's Conjecture

In this section, we give a generalization of the Fibonomial coefficients in order to state a general version of the conjecture of Melham.

Definition 1. *For all integers n, m, r with $m, r \geq 1$, the r–Fibonomial is defined as*

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r := \frac{F_n F_{n-r} \dots F_{n-r(m-1)}}{F_r F_{2r} \dots F_{mr}}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_r = 1$ where F_n is the nth Fibonacci number.

Now we give a generalization of the conjecture of Melham in terms of r–Fibonomials:

Theorem 1. *For any integers m, n and k:*

(i) *If r odd, then*

$$\sum_{j=0}^{m-1}(-1)^{\frac{j(j-1)}{2}}\begin{Bmatrix}(m+1)k+mr\\ j\end{Bmatrix}_r\begin{Bmatrix}(m+1)k+r(m-j-1)\\ m-j-1\end{Bmatrix}_r F_{n+k+r(m-j)}^{m+1}$$
$$+(-1)^{\frac{m(m-1)}{2}}F_{n-mk}^{m+1}=\left(\prod_{j=1}^{m}F_{(m+1)k+rj}\right)F_{(m+1)(n+\frac{rm}{2})}.$$

(ii) *If r even, then*

$$\sum_{j=0}^{m-1}(-1)^{j}\begin{Bmatrix}(m+1)k+mr\\ j\end{Bmatrix}_r\begin{Bmatrix}(m+1)k+r(m-j-1)\\ m-j-1\end{Bmatrix}_r F_{n+k+r(m-j)}^{m+1}$$
$$+(-1)^{m}F_{n-mk}^{m+1}=\left(\prod_{j=1}^{m}F_{(m+1)k+rj}\right)F_{(m+1)(n+\frac{rm}{2})}.$$

(iii) *The Lucas counterpart of (i),*

$$\sum_{j=0}^{m-1}(-1)^{\frac{j(j-1)}{2}}\begin{Bmatrix}(m+1)k+mr\\ j\end{Bmatrix}_r\begin{Bmatrix}(m+1)k+r(m-j-1)\\ m-j-1\end{Bmatrix}_r \times$$
$$L_{n+k+r(m-j)}^{m+1}+(-1)^{\frac{m(m-1)}{2}}L_{n-mk}^{m+1}$$
$$=\begin{cases}5^{\frac{m+1}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+rj}\right)F_{(m+1)(n+\frac{rm}{2})} & \text{if } m \text{ is odd,}\\ 5^{\frac{m}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+rj}\right)L_{(m+1)(n+\frac{rm}{2})} & \text{if } m \text{ is even.}\end{cases}$$

(iv) *The Lucas counterpart of (ii),*

$$\sum_{j=0}^{m-1}(-1)^{j}\begin{Bmatrix}(m+1)k+mr\\ j\end{Bmatrix}_r\begin{Bmatrix}(m+1)k+r(m-j-1)\\ m-j-1\end{Bmatrix}_r L_{n+k+r(m-j)}^{m+1}$$
$$+(-1)^{m}L_{n-mk}^{m+1}=\begin{cases}5^{\frac{m+1}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+rj}\right)F_{(m+1)(n+\frac{rm}{2})} & \text{if } m \text{ is odd,}\\ 5^{\frac{m}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+rj}\right)L_{(m+1)(n+\frac{rm}{2})} & \text{if } m \text{ is even.}\end{cases}$$

Proof. We want to point out that rewriting (i)–(iv) in terms of q–binomials produces the same expression for (i) and (ii) and also for (iii) and (iv). We can combine all the cases (i)–(iv) in one formula as follows:

$$(1-q^{(m+1)k+rm})\frac{(q^{(m+1)k+r};q^r)_{m-1}}{(q^r;q^r)_{m-1}}\times$$
$$\sum_{j=0}^{m-1}(-1)^j q^{\frac{rj(j+1)}{2}}\begin{bmatrix}m-1\\ j\end{bmatrix}_{q^r}\frac{(1+(-1)^h q^{n+k+r(m-j)})^{m+1}}{1-q^{(m+1)k+r(m-j)}}$$
$$=\left(1-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}\right)(q^{(m+1)k+r};q^r)_m$$
$$-(-1)^m q^{\frac{m(m+1)(2k+r)}{2}}(1+(-1)^h q^{n-mk})^{m+1}. \tag{2.1}$$

Here, $h=1$ in (2.1) gives us the q–notation of the cases (i), (ii) and similarly $h=0$ in (2.1) gives the cases (iii) and (iv). For the proof, let us consider

$$X := \frac{1}{(1-z)(1-zq^r)\dots(1-zq^{r(m-1)})}\frac{\big(1+(-1)^h zq^{n+k+rm}\big)^{m+1}}{z(1-zq^{(m+1)k+rm})}.$$

Performing *partial fraction decomposition*, we get:

$$X = \sum_{j=0}^{m-1}\frac{(-1)^j q^{r\binom{j+1}{2}}}{(q^r;q^r)_j(q^r;q^r)_{m-1-j}}\frac{\big(1+(-1)^h q^{n+k+rm-rj}\big)^{m+1}}{1-q^{(m+1)k+rm-rj}}\frac{1}{q^{-rj}(1-zq^{rj})}$$
$$+\frac{C}{1-zq^{(m+1)k+rm}}+\frac{1}{z}$$

with

$$C = \frac{\big(1+(-1)^h zq^{n+k+rm}\big)^{m+1}}{z(1-z)(1-zq^r)\dots(1-zq^{r(m-1)})}\Bigg|_{z=q^{-(m+1)k-rm}}.$$

We note that the degree of numerator and denominator is $m+1$. As $z\to\infty$,

$$X \sim \frac{B}{z}+O(z^{-2})$$

with

$$B = \frac{(-1)^m}{q^{r\binom{m}{2}}}\frac{\big((-1)^h q^{n+k+rm}\big)^{m+1}}{-q^{(m+1)k+rm}} = (-1)^{(h+1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}.$$

Furthermore,

$$zX = \sum_{j=0}^{m-1}\frac{(-1)^j q^{r\binom{j+1}{2}}}{(q^r;q^r)_j(q^r;q^r)_{m-1-j}}\frac{\big(1+(-1)^h q^{n+k+rm-rj}\big)^{m+1}}{1-q^{(m+1)k+rm-rj}}\times$$
$$\frac{z}{q^{-rj}(1-zq^{rj})}+\frac{Cz}{1-zq^{(m+1)k+rm}}+1,$$

and letting z approach ∞:

$$B = \sum_{j=0}^{m-1}\frac{(-1)^{j-1} q^{r\binom{j+1}{2}}}{(q^r;q^r)_j(q^r;q^r)_{m-1-j}}\frac{\big(1+(-1)^h q^{n+k+rm-rj}\big)^{m+1}}{1-q^{(m+1)k+rm-rj}}$$
$$-\frac{C}{q^{(m+1)k+rm}}+1.$$

Thus, we get:

$$(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^r;q^r)_{m-1}$$
$$=\sum_{j=0}^{m-1}(-1)^{j-1}q^{r\binom{j+1}{2}}\begin{bmatrix}m-1\\ j\end{bmatrix}_{q^r}\frac{\big(1+(-1)^h q^{n+k+r(m-j)}\big)^{m+1}}{1-q^{(m+1)k+r(m-j)}}$$
$$-C\cdot\frac{(q^r;q^r)_{m-1}}{q^{(m+1)k+rm}}+(q^r;q^r)_{m-1},$$

or

$$\sum_{j=0}^{m-1}(-1)^j q^{r\binom{j+1}{2}}\begin{bmatrix}m-1\\ j\end{bmatrix}_{q^r}\frac{\big(1+(-1)^h q^{n+k+r(m-j)}\big)^{m+1}}{1-q^{(m+1)k+r(m-j)}}$$
$$= -C\cdot\frac{(q^r;q^r)_{m-1}}{q^{(m+1)k+rm}}+(q^r;q^r)_{m-1}-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^r;q^r)_{m-1}. \tag{2.2}$$

Now we work out the constant C:

$$C=\frac{1}{q^{-(m+1)k-rm}}\times$$
$$\frac{\big(1+(-1)^h q^{n-mk}\big)^{m+1}}{(1-q^{-(m+1)k-rm})(1-q^{-(m+1)k-rm}q^r)\dots(1-q^{-(m+1)k-rm}q^{r(m-1)})}$$
$$=\frac{(-1)^m\big(1+(-1)^h q^{n-mk}\big)^{m+1}q^{((m+1)k+rm)m-r\binom{m}{2}}}{q^{-(m+1)k-rm}(1-q^{(m+1)k+rm})(1-q^{(m+1)k+rm-r})\dots(1-q^{(m+1)k+r})}$$
$$=\frac{(-1)^m\big(1+(-1)^h q^{n-mk}\big)^{m+1}q^{((m+1)k+rm)(m+1)-r\binom{m}{2}}}{(1-q^{(m+1)k+rm})(1-q^{(m+1)k+rm-r})\dots(1-q^{(m+1)k+r})}$$
$$=\frac{(-1)^m\big(1+(-1)^h q^{n-mk}\big)^{m+1}q^{(m+1)^2k+rm(m+1)-r\binom{m}{2}}}{(q^{(m+1)k+r};q^r)_m}. \tag{2.3}$$

Using (2.2) and (2.3), we obtain

$$\sum_{j=0}^{m-1}(-1)^j q^{r\binom{j+1}{2}}\begin{bmatrix}m-1\\ j\end{bmatrix}_{q^r}\frac{\big(1+(-1)^h q^{n+k+r(m-j)}\big)^{m+1}}{1-q^{(m+1)k+r(m-j)}}$$
$$=-\frac{(-1)^m(1+(-1)^h q^{n-mk})^{m+1}q^{(m+1)mk+r\binom{m+1}{2}}}{(q^{(m+1)k+r};q^r)_m}\cdot(q^r;q^r)_{m-1}$$
$$+(q^r;q^r)_{m-1}-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^r;q^r)_{m-1}.$$

Now we show that this is equivalent to (2.1). We write (2.1), but replace the sum by the formula that we just obtained and get

$$(1-q^{(m+1)k+rm})\frac{(q^{(m+1)k+r};q^r)_{m-1}}{(q^r;q^r)_{m-1}}\times$$
$$\Bigg[-\frac{(-1)^m\big(1+(-1)^h q^{n-mk}\big)^{m+1}q^{(m+1)mk+r\binom{m+1}{2}}}{(q^{(m+1)k+r};q^r)_m}\cdot(q^r;q^r)_{m-1}$$
$$+(q^r;q^r)_{m-1}-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^r;q^r)_{m-1}\Bigg]$$
$$=\Big(1-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}\Big)(q^{(m+1)k+r};q^r)_m$$
$$-(-1)^m q^{\frac{m(m+1)(2k+r)}{2}}\big(1+(-1)^h q^{n-mk}\big)^{m+1}.$$

Our goal is achieved once we see that this is an identity. For that, we will gradually simplify it until a trivial identity remains.

We need to prove that

$$(1-q^{(m+1)k+rm})(q^{(m+1)k+r};q^r)_{m-1}\times$$
$$\Bigg[-\frac{(-1)^m\big(1+(-1)^hq^{n-mk}\big)^{m+1}q^{(m+1)mk+r\binom{m+1}{2}}}{(q^{(m+1)k+r};q^r)_m}$$
$$+1-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}\Bigg]$$
$$=\Big(1-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}\Big)(q^{(m+1)k+r};q^r)_m$$
$$-(-1)^mq^{\frac{m(m+1)(2k+r)}{2}}\big(1+(-1)^hq^{n-mk}\big)^{m+1}.$$

Combining the first two factors,

$$(q^{(m+1)k+r};q^r)_m\times$$
$$\Bigg[-\frac{(-1)^m\big(1+(-1)^hq^{n-mk}\big)^{m+1}q^{(m+1)mk+r\binom{m+1}{2}}}{(q^{(m+1)k+r};q^r)_m}$$
$$+1-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}\Bigg]$$
$$=\Big(1-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}\Big)(q^{(m+1)k+r};q^r)_m$$
$$-(-1)^mq^{\frac{m(m+1)(2k+r)}{2}}\big(1+(-1)^hq^{n-mk}\big)^{m+1},$$

which we further rewrite:

$$(-1)^{m-1}\big(1+(-1)^hq^{n-mk}\big)^{m+1}q^{(m+1)mk+r\binom{m+1}{2}}+(q^{(m+1)k+r};q^r)_m$$
$$-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^{(m+1)k+r};q^r)_m$$
$$=\Big(1-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}\Big)(q^{(m+1)k+r};q^r)_m$$
$$+(-1)^{m-1}q^{\frac{m(m+1)(2k+r)}{2}}\big(1+(-1)^hq^{n-mk}\big)^{m+1},$$

or

$$(q^{(m+1)k+r};q^r)_m-(-1)^{(h+1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^{(m+1)k+r};q^r)_m$$
$$=\Big(1-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}\Big)(q^{(m+1)k+r};q^r)_m\ ,$$

or

$$-(-1)^{(h-1)(m+1)}q^{n(m+1)+r\binom{m+1}{2}}(q^{(m+1)k+r};q^r)_m$$
$$=-(-1)^{(h-1)(m+1)}q^{\frac{(m+1)(2n+rm)}{2}}(q^{(m+1)k+r};q^r)_m.$$

Finally, we see that this is correct, and so we have proved the identities (i)–(iv). ∎

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[1]TOBB University of Economics and Technology Mathematics Department 06560 Sögütözü Ankara Turkey
E-mail address: ekilic@etu.edu.tr[1]

[2]Ankara University Mathematics Department 06100 Ankara Turkey
E-mail address: iakkus@science.ankara.edu.tr[2]

[3]Department of Mathematics, University of Stellenbosch 7602 Stellenbosch South Africa
E-mail address: hproding@sun.ac.za[3]