PARTIAL FRACTION DECOMPOSITION PROOFS OF SOME q-SERIES IDENTITIES

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Abstract. Using the partial fraction decomposition method, we give new proofs of some q-series identities related to divisor functions in [10] and two finite q-series transformations in [8]. Likewise, some new q-series identities are obtained, including generalizations of some main results in [10] and generalizations of special cases of the q-Pfaff-Saalschütz summation theorem and the q-Chu-Vandermonde identity.

1. Introduction

Wenchang Chu [2] showed that some seemingly difficult identities can be proved in a simple way by performing partial fraction decomposition to a suitable rational function, and then taking a certain limit. For further applications of the partial fraction decomposition method, see [1, 3–5].

Although for the cases that we study here, the method called q-Rice method [11] is computationally equivalent, we prefer to express everything in terms of partial fraction decomposition, which is conceptually simpler than contour integrals and residues.

We reprove in this way some identities from [10] and from [8] in this very simple fashion and obtain also some additional formulæ. Especially, we generalize some main results in [10] and special cases of the q-Pfaff-Saalschütz summation theorem and the q-Chu-Vandermonde identity.

As usual, we follow the notation and terminology in [9]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \text{ for } n \in \mathbb{C}.$$

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or infinity.

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The q-binomial coefficients, or the Gauss coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} z^k.$$

2. Some q-series identities of Guo-Zhang

In [10], the authors proved new generalizations of some q-series identities of Dilcher [7] and Prodinger [11] related to divisor functions. They also obtained some special cases including an identity related to overpartitions given by Corteel and Lovejoy [6, Theorem 4.4].

In this section, using the partial fraction decomposition method, we give new proofs of some theorems in [10].

Theorem 2.1. *[10, Theorem 1.1] For $n \ge 0$ and $0 \le l, m \le n$, we have*

$$\sum_{\substack{k=0 \\ k \neq m}}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q/t;q)_k (tq^{-l};q)_{n-k}}{1-q^{k-m}} t^k = (-1)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix} (tq^{-l};q)_l (tq^{-m};q)_{n-l}$$
$$\times \left(\sum_{k=0}^{n-l-1} \frac{tq^{k-m}}{1-tq^{k-m}} - \sum_{\substack{k=0 \\ k \neq m}}^{n} \frac{q^{k-m}}{1-q^{k-m}} \right).$$

Proof. Set

$$F(z) = \frac{(q;q)_n}{(z;q)_{n+1}} \frac{(tz;q)_{n-l} (tq^{-l};q)_l}{(z-q^{-m})}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=0}^{n} \frac{b_k}{1-zq^k} + \frac{b_{n+1}}{(1-zq^m)^2}. \tag{2.1}$$

Multiplying both sides of (2.1) by z, and then letting $z \to \infty$, we get

$$\sum_{k=0}^{n} \frac{b_k}{q^k} = 0. \tag{2.2}$$

Now we compute b_k for $k = 0, 1, \ldots, n$.

For $k \neq m$, we have

$$b_k = \lim_{z \to q^{-k}} (1-zq^k) F(z)$$

$$
\begin{aligned}
&= \lim_{z\to q^{-k}} (1-zq^k)\frac{(q;q)_n}{(z;q)_{n+1}}\frac{(tz;q)_{n-l}(tq^{-l};q)_l}{(z-q^{-m})} \\
&= \lim_{z\to q^{-k}} \frac{(q;q)_n(tz;q)_{n-l}(tq^{-l};q)_l}{(z;q)_k(zq^{k+1};q)_{n-k}(z-q^{-m})} \\
&= \frac{(q;q)_n(tq^{-k};q)_{n-l}(tq^{-l};q)_l}{(q^{-k};q)_k(q;q)_{n-k}(q^{-k}-q^{-m})} \\
&= \frac{(-1)^k q^{\binom{k+1}{2}+k}(q;q)_n(tq^{-k};q)_\infty(tq^{-l};q)_\infty}{(tq^{-k+n-l};q)_\infty(t;q)_\infty(q;q)_k(q;q)_{n-k}(1-q^{k-m})} \\
&= \begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{\binom{k+1}{2}+k}(tq^{-k};q)_k(tq^{-l};q)_{n-k}}{1-q^{k-m}} \\
&= \begin{bmatrix} n \\ k \end{bmatrix}\frac{(q/t;q)_k(tq^{-l};q)_{n-k}}{1-q^{k-m}}(tq)^k.
\end{aligned}
$$

For $k=m$, we have

$$
\begin{aligned}
b_m &= -q^{-m}\lim_{z\to q^{-m}} D_z\{(1-zq^m)^2F(z)\} \\
&= -q^{-m}\lim_{z\to q^{-m}} D_z\left\{(1-zq^m)^2\frac{(q;q)_n}{(z;q)_{n+1}}\frac{(tz;q)_{n-l}(tq^{-l};q)_l}{(z-q^{-m})}\right\} \\
&= \lim_{z\to q^{-m}} D_z\left\{\frac{(q;q)_n(tz;q)_{n-l}(tq^{-l};q)_l}{(z;q)_m(zq^{m+1};q)_{n-m}}\right\} \\
&= (q;q)_n(tq^{-l};q)_l\lim_{z\to q^{-m}} D_z\left\{\frac{(tz;q)_{n-l}}{(z;q)_m(zq^{m+1};q)_{n-m}}\right\} \\
&= \frac{(q;q)_n(tq^{-l};q)_l(tq^{-m};q)_{n-l}}{(-1)^m q^{-\binom{m+1}{2}}(q;q)_m(q;q)_{n-m}}\left(-\sum_{k=0}^{n-l-1}\frac{tq^k}{1-tq^{k-m}}+\sum_{\substack{k=0\\k\neq m}}^{n}\frac{q^k}{1-q^{k-m}}\right) \\
&= (-1)^{m-1}q^{\binom{m+1}{2}}\begin{bmatrix} n \\ m \end{bmatrix}(tq^{-l};q)_l(tq^{-m};q)_{n-l}\left(\sum_{k=0}^{n-l-1}\frac{tq^k}{1-tq^{k-m}}-\sum_{\substack{k=0\\k\neq m}}^{n}\frac{q^k}{1-q^{k-m}}\right).
\end{aligned}
$$

Therefore, according to (2.2), we get Theorem 2.1. □

Theorem 2.2. *[10, Theorem 1.2] For $m,n\geq 1$, we have*

$$
\begin{aligned}
&\sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(q^m/t;q)_k(t;q)_{n-k}}{(tq^{-m};q)_{m+n}(1-q^k)^m}t^k \\
&\qquad = -\sum_{1\leq k_m\leq\cdots\leq k_1\leq n}\frac{q^{k_1}}{(1-tq^{k_1-1})(1-q^{k_1})}\cdots\frac{q^{k_m}}{(1-tq^{k_m-m})(1-q^{k_m})}.
\end{aligned}
$$

Proof. Set

$$
F(z)=\frac{(q;q)_n}{(z;q)_{n+1}}\frac{(tzq^{-m+1};q)_{m+n-1}}{(tq^{-m};q)_{m+n}(tq^{-m+1};q)_{m-1}(z-1)^m}.
$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z)=\sum_{k=0}^{n}\frac{b_k}{1-zq^k}+\sum_{k=2}^{m+1}\frac{c_k}{(1-z)^k}. \tag{2.3}$$

Multiplying both sides of (2.3) by z, and then letting $z\to\infty$, we get

$$\sum_{k=0}^{n}\frac{b_k}{q^k}=0. \tag{2.4}$$

Now we calculate b_k for $0\le k\le n$.

For $1\le k\le n$, we have

$$\begin{aligned}
b_k &= \lim_{z\to q^{-k}}(1-zq^k)F(z)\\
&= \lim_{z\to q^{-k}}\frac{(q;q)_n(tzq^{-m+1};q)_{m+n-1}}{(z;q)_k(zq^{k+1};q)_{n-k}(tq^{-m};q)_{m+n}(tq^{-m+1};q)_{m-1}(z-1)^m}\\
&= \frac{(q;q)_n(tq^{-m-k+1};q)_{m+n-1}}{(q^{-k};q)_k(q;q)_{n-k}(tq^{-m};q)_{m+n}(tq^{-m+1};q)_{m-1}(q^{-k}-1)^m}\\
&= \begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^kq^{\binom{k+1}{2}+km}(tq^{-m-k+1};q)_{m+n-1}}{(tq^{-m};q)_{m+n}(tq^{-m+1};q)_{m-1}(1-q^k)^m}\\
&= \begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^kq^{\binom{k+1}{2}+km}(tq^{-m-k+1};q)_k(t;q)_{n-k}}{(tq^{-m};q)_{m+n}(1-q^k)^m}\\
&= \begin{bmatrix}n\\k\end{bmatrix}\frac{(q^m/t;q)_k(t;q)_{n-k}}{(tq^{-m};q)_{m+n}(1-q^k)^m}(tq)^k.
\end{aligned}$$

For $k=0$, we have

$$\begin{aligned}
b_0 &= \frac{(-1)^m}{m!}\lim_{z\to1}D_z^{(m)}\left\{(1-z)^{m+1}F(z)\right\}\\
&= \frac{(-1)^m}{m!}\lim_{z\to1}D_z^{(m)}\left\{\frac{(-1)^m(q;q)_n(tzq^{-m+1};q)_{m+n-1}}{(zq;q)_n(tq^{-m};q)_{m+n}(tq^{-m+1};q)_{m-1}}\right\}\\
&= \frac{(q;q)_n}{(tq^{-m};q)_{m+n}(tq^{-m+1};q)_{m-1}m!}\lim_{z\to1}D_z^{(m)}\left\{\frac{(tzq^{-m+1};q)_{m+n-1}}{(zq;q)_n}\right\}\\
&= \frac{(q;q)_n}{(tq^{-m};q)_m(tq^{-m+1};q)_{m+n-1}}[(z-1)^m]\frac{(tzq^{-m+1};q)_{m+n-1}}{(zq;q)_n}\\
&= \frac{(q;q)_n}{(tq^{-m};q)_m(tq^{-m+1};q)_{m+n-1}}[w^m]\frac{(t(w+1)q^{-m+1};q)_{m+n-1}}{((w+1)q;q)_n}\\
&= \frac{1}{(tq^{-m};q)_m}[w^m]\frac{\prod_{k=1}^{m+n-1}\left(1-\frac{twq^{-m+k}}{1-tq^{-m+k}}\right)}{\prod_{k=1}^{n}\left(1-\frac{wq^k}{1-q^k}\right)}.
\end{aligned}$$

In order to read off this coefficient, we have

$$
\begin{aligned}
&[w^m]\frac{\prod_{k=1-m}^{n-1}\left(1-\frac{twq^k}{1-tq^k}\right)}{\prod_{k=1}^{n}\left(1-\frac{wq^k}{1-q^k}\right)}\\
&=\sum_{1\le k_m\le\cdots\le k_1\le n}\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-1}}{1-tq^{k_1-1}}\right)\cdots\left(\frac{q^{k_m}}{1-q^{k_m}}-\frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right)\\
&=\sum_{1\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}(1-tq^{-1})}{(1-tq^{k_1-1})(1-q^{k_1})}\cdots\frac{q^{k_m}(1-tq^{-m})}{(1-tq^{k_m-m})(1-q^{k_m})}\\
&=(tq^{-m};q)_m\sum_{1\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}}{(1-tq^{k_1-1})(1-q^{k_1})}\cdots\frac{q^{k_m}}{(1-tq^{k_m-m})(1-q^{k_m})}.
\end{aligned}
$$

For the first equation of this chain of equalities, we argue as follows: We have to select altogether m w's from the factors. Factors in the denominator can be chosen arbitrary often, but factors from the numerator only once. Let k_1 be the largest number such that k_1-1 from the numerator is chosen or, if not, such that k_1 is chosen from the denominator. This gives a contribution

$$\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-1}}{1-tq^{k_1-1}}\right).$$

Now let $k_2\le k_1$ be the largest number such that k_2-2 from the numerator is chosen or, if not, such that k_2 is chosen from the denominator. This gives a contribution

$$\left(\frac{q^{k_2}}{1-q^{k_2}}-\frac{tq^{k_2-2}}{1-tq^{k_2-2}}\right).$$

Then let $k_3\le k_2$ be the largest number such that k_3-3 from the numerator is chosen or, if not, such that k_3 is chosen from the denominator, and so on.

Therefore, we have

$$b_0=\sum_{1\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}}{(1-tq^{k_1-1})(1-q^{k_1})}\cdots\frac{q^{k_m}}{(1-tq^{k_m-m})(1-q^{k_m})}.$$

According to (2.4), we get Theorem 2.2. □

Theorem 2.3. *[10, Theorem 4.1] For $m,n\ge 1$, we have*

$$
\begin{aligned}
&\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q^m/t;q)_k(t/q;q)_{n-k}}{(tq^{-m};q)_{m+n-1}(1-q^k)^m}t^k\\
&\quad=\sum_{k_1=1}^{n}\frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots\\
&\quad\quad\times\left(\sum_{k_m=1}^{k_{m-1}-1}\frac{tq^{k_m-m}}{1-tq^{k_m-m}}-\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}\right).
\end{aligned}
$$

Proof. Set

$$F(z)=\frac{(q;q)_n}{(z;q)_{n+1}}\frac{(tzq^{-m+1};q)_{m+n-2}}{(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}(z-1)^m}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z)=\sum_{k=0}^{n}\frac{b_k}{1-zq^k}+\sum_{k=2}^{m+1}\frac{c_k}{(1-z)^k}. \tag{2.5}$$

Multiplying by z on both sides of (2.5), and letting $z\to\infty$, we get

$$\sum_{k=0}^{n}\frac{b_k}{q^k}=0. \tag{2.6}$$

Now we calculate b_k for $0\le k\le n$.

For $1\le k\le n$, we have

$$\begin{aligned}
b_k&=\lim_{z\to q^{-k}}(1-zq^k)F(z)\\
&=\lim_{z\to q^{-k}}\frac{(q;q)_n(tzq^{-m+1};q)_{m+n-2}}{(z;q)_k(zq^{k+1};q)_{n-k}(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}(z-1)^m}\\
&=\frac{(q;q)_n(tq^{-m-k+1};q)_{m+n-2}}{(q^{-k};q)_k(q;q)_{n-k}(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}(q^{-k}-1)^m}\\
&=\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^kq^{\binom{k+1}{2}+km}(tq^{-m-k+1};q)_{m+n-2}}{(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}(1-q^k)^m}\\
&=\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^kq^{\binom{k+1}{2}+km}(tq^{-m-k+1};q)_k(t/q;q)_{n-k}}{(tq^{-m};q)_{m+n-1}(1-q^k)^m}\\
&=\begin{bmatrix}n\\k\end{bmatrix}\frac{(q^m/t;q)_k(t/q;q)_{n-k}}{(tq^{-m};q)_{m+n-1}(1-q^k)^m}(tq)^k.
\end{aligned}$$

For $k=0$, we have

$$\begin{aligned}
b_0&=\frac{(-1)^m}{m!}\lim_{z\to1}D_z^{(m)}\left\{(1-z)^{m+1}F(z)\right\}\\
&=\frac{(-1)^m}{m!}\lim_{z\to1}D_z^{(m)}\left\{\frac{(-1)^m(q;q)_n(tzq^{-m+1};q)_{m+n-2}}{(zq;q)_n(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}}\right\}\\
&=\frac{(q;q)_n}{(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}m!}\lim_{z\to1}D_z^{(m)}\left\{\frac{(tzq^{-m+1};q)_{m+n-2}}{(zq;q)_n}\right\}\\
&=\frac{(q;q)_n}{(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}}[(z-1)^m]\frac{(tzq^{-m+1};q)_{m+n-2}}{(zq;q)_n}\\
&=\frac{(q;q)_n}{(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}}[w^m]\frac{(t(w+1)q^{-m+1};q)_{m+n-2}}{((w+1)q;q)_n}
\end{aligned}$$

$$= \frac{1}{(tq^{-m};q)_{m-1}}[w^m]\frac{\prod_{k=1}^{m+n-2}\left(1-\frac{twq^{-m+k}}{1-tq^{-m+k}}\right)}{\prod_{k=1}^{n}\left(1-\frac{wq^k}{1-q^k}\right)}.$$

In order to read off this coefficient, we have

$$\begin{aligned}
&[w^m]\frac{\prod_{k=1-m}^{n-2}\left(1-\frac{twq^k}{1-tq^k}\right)}{\prod_{k=1}^{n}\left(1-\frac{wq^k}{1-q^k}\right)} \\
&= \sum_{k_1=1}^{n}\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-2}}{1-tq^{k_1-2}}\right)\sum_{k_2=1}^{k_1}\left(\frac{q^{k_2}}{1-q^{k_2}}-\frac{tq^{k_2-3}}{1-tq^{k_2-3}}\right)\cdots \\
&\qquad \times \sum_{k_{m-1}=1}^{k_{m-2}}\left(\frac{q^{k_{m-1}}}{1-q^{k_{m-1}}}-\frac{tq^{k_{m-1}-m}}{1-tq^{k_{m-1}-m}}\right) \\
&\qquad\qquad \times \sum_{k_m=1}^{k_{m-1}}\left[\left(\frac{q^{k_m}}{1-q^{k_m}}-\frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}}\right)-\left(-\frac{tq^{-m}}{1-tq^{-m}}\right)\right] \\
&= \sum_{k_1=1}^{n}\frac{q^{k_1}(1-tq^{-2})}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}(1-tq^{-3})}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots \\
&\qquad \times \sum_{k_{m-1}=1}^{k_{m-2}}\frac{q^{k_{m-1}}(1-tq^{-m})}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}})} \\
&\qquad\qquad \times \left(\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}-\sum_{k_m=1}^{k_{m-1}}\frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}}+\frac{tq^{-m}}{1-tq^{-m}}\right) \\
&= (tq^{-m};q)_{m-1}\sum_{k_1=1}^{n}\frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots \\
&\qquad \times \sum_{k_{m-1}=1}^{k_{m-2}}\frac{q^{k_{m-1}}}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}})}\left(\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}-\sum_{k_m=2}^{k_{m-1}}\frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}}\right) \\
&= (tq^{-m};q)_{m-1}\sum_{k_1=1}^{n}\frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots \\
&\qquad \times \sum_{k_{m-1}=1}^{k_{m-2}}\frac{q^{k_{m-1}}}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}})}\left(\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}-\sum_{k_m=1}^{k_{m-1}-1}\frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right).
\end{aligned}$$

The explanation is very similar to the earlier instance Theorem 2.2. The difference is that, if $k_m = 1$, there is an exception, since $k_m - m - 1 = -m$ is out of range for the numerator, and thus such a term cannot be taken.

Perhaps a more aesthetic way would be to write

$$[w^m]\frac{\prod_{k=1-m}^{n-2}\left(1-\frac{twq^k}{1-tq^k}\right)}{\prod_{k=1}^{n}\left(1-\frac{wq^k}{1-q^k}\right)}$$

$$= \sum_{2\le k_m\le\cdots\le k_1\le n}\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-2}}{1-tq^{k_1-2}}\right)\cdots\left(\frac{q^{k_m}}{1-q^{k_m}}-\frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}}\right)$$

$$+ \sum_{1\le k_{m-1}\le\cdots\le k_1\le n}\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-2}}{1-tq^{k_1-2}}\right)\cdots\left(\frac{q^{k_{m-1}}}{1-q^{k_{m-1}}}-\frac{tq^{k_{m-1}-m}}{1-tq^{k_{m-1}-m}}\right)\frac{q}{1-q}$$

$$= \sum_{2\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}(1-tq^{-2})}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_m}(1-tq^{-m-1})}{(1-q^{k_m})(1-tq^{k_m-m-1})}$$

$$+ \sum_{1\le k_{m-1}\le\cdots\le k_1\le n}\frac{q^{k_1}(1-tq^{-2})}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_{m-1}}(1-tq^{-m})}{(1-q^{k_{m-1}})(1-tq^{k_{m-1}-m})}\frac{q}{1-q}$$

$$= (tq^{-m-1};q)_m\sum_{2\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_m}}{(1-q^{k_m})(1-tq^{k_m-m-1})}$$

$$+ (tq^{-m};q)_{m-1}\sum_{1\le k_{m-1}\le\cdots\le k_1\le n}\frac{q^{k_1}}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_{m-1}}}{(1-q^{k_{m-1}})(1-tq^{k_{m-1}-m})}\frac{q}{1-q}.$$

Therefore, we have

$$b_0 = \sum_{k_1=1}^{n}\frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots$$
$$\times\left(\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}-\sum_{k_m=1}^{k_{m-1}-1}\frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right).$$

According to (2.6), we get Theorem 2.3. □

3. New results

In this section, using the partial fraction decomposition method, we get new generalizations of some main results in [10], and also obtain some other new q-series identities, including generalizations of special cases of the q-Pfaff-Saalschütz summation theorem and the q-Chu-Vandermonde identity.

After that, we give proofs of three theorems given by Guo and Zhang in [10]. While we don't find a proper way to prove [10, Theorem 1.3] by using the partial fraction decomposition method. But we get the following similar new result.

Theorem 3.1. *For $m,n\ge 1$, we have*

$$\sum_{k=1}^{n}\frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}t^k-\sum_{k=1}^{m}\frac{(tq^k;q)_{m-k}(t;q)_{n+k}(q/v;q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}v^k$$

$$= \frac{(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k(1-v/q)}{(1-vq^{k-1})(1-q^k)}-\sum_{k=1}^{n}\frac{q^k(1-t/q)}{(1-tq^{k-1})(1-q^k)}\right). \quad (3.1)$$

Proof. Set

$$F(z)=-\frac{(tz;q)_n(t;q)_m(zq^{1-m}/v;q)_m}{(z;q)_{n+1}(v;q)_m(zq^{-m};q)_{m+1}}\left(\frac{v}{q}\right)^m.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z)=\sum_{k=1}^{n}\frac{b_k}{1-zq^k}+\sum_{k=1}^{m}\frac{c_k}{1-zq^{-k}}+\frac{a_1}{1-z}+\frac{a_2}{(1-z)^2}. \quad (3.2)$$

Multiplying by z on both sides of (3.2), and letting $z\to\infty$, we get

$$\sum_{k=1}^{n}\frac{b_k}{q^k}+\sum_{k=1}^{m}\frac{c_k}{q^{-k}}=-a_1. \quad (3.3)$$

For $1\le k\le n$, we have

$$\begin{aligned}
b_k &= \lim_{z\to q^{-k}}(1-zq^k)F(z)\\
&=-\lim_{z\to q^{-k}}\frac{(tz;q)_n(t;q)_m(zq^{1-m}/v;q)_m}{(z;q)_k(zq^{k+1};q)_{n-k}(v;q)_m(zq^{-m};q)_{m+1}}\left(\frac{v}{q}\right)^m\\
&=-\frac{(tq^{-k};q)_n(t;q)_m(q^{-m-k+1}/v;q)_m}{(q^{-k};q)_k(q;q)_{n-k}(v;q)_m(q^{-m-k};q)_{m+1}}\left(\frac{v}{q}\right)^m\\
&=\frac{(-1)^kq^{\binom{k+1}{2}+k}(tq^{-k};q)_n(t;q)_m(vq^k;q)_m}{(q;q)_k(q;q)_{n-k}(v;q)_m(q^k;q)_{m+1}}\\
&=\frac{(-1)^kq^{\binom{k+1}{2}+k}(tq^{-k};q)_k(t;q)_m(t;q)_{n-k}(vq^m;q)_k}{(q;q)_k(q;q)_{n-k}(v;q)_k(q^k;q)_{m+1}}\\
&=\frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}(tq)^k.
\end{aligned}$$

For $1\le k\le m$, we have

$$\begin{aligned}
c_k &= \lim_{z\to q^{k}}(1-zq^{-k})F(z)\\
&=-\lim_{z\to q^{k}}\frac{(tz;q)_n(t;q)_m(zq^{1-m}/v;q)_m}{(z;q)_{n+1}(v;q)_m(zq^{-m};q)_{m-k}(zq^{-k+1};q)_k}\left(\frac{v}{q}\right)^m\\
&=-\frac{(tq^k;q)_n(t;q)_m(q^{-m+k+1}/v;q)_m}{(q^k;q)_{n+1}(v;q)_m(q^{-m+k};q)_{m-k}(q;q)_k}\left(\frac{v}{q}\right)^m\\
&=\frac{(-1)^{k+1}q^{\binom{k}{2}}(tq^k;q)_n(t;q)_m(vq^{-k};q)_m}{(q^k;q)_{n+1}(v;q)_m(q;q)_{m-k}(q;q)_k}\\
&=\frac{(-1)^{k+1}q^{\binom{k}{2}}(tq^k;q)_{m-k}(t;q)_{n+k}(vq^{-k};q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}
\end{aligned}$$

$$= -\frac{(tq^k;q)_{m-k}(t;q)_{n+k}(q/v;q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}\left(\frac{v}{q}\right)^k.$$

Now we compute a_1.

$$\begin{aligned}
a_1 &= -\lim_{z\to1} D_z\{(1-z)^2F(z)\}\\
&= \lim_{z\to1} D_z\left\{\frac{(tz;q)_n(t;q)_m(zq^{1-m}/v;q)_m}{(zq;q)_n(v;q)_m(zq^{-m};q)_m}\left(\frac{v}{q}\right)^m\right\}\\
&= \frac{(t;q)_m}{(v;q)_m}\left(\frac{v}{q}\right)^m \lim_{z\to1} D_z\left\{\frac{(tz;q)_n(zq^{1-m}/v;q)_m}{(zq;q)_n(zq^{-m};q)_m}\right\}\\
&= \frac{(t;q)_m}{(v;q)_m}\left(\frac{v}{q}\right)^m \lim_{z\to1}\Bigg[\frac{(tz;q)_n(zq^{1-m}/v;q)_m}{(zq;q)_n(zq^{-m};q)_m}\\
&\qquad\times\left(-\sum_{k=0}^{n-1}\frac{tq^k}{1-tzq^k}+\sum_{k=0}^{m-1}\frac{1}{z-vq^k}+\sum_{k=1}^{n}\frac{q^k}{1-zq^k}-\sum_{k=1}^{m}\frac{1}{z-q^k}\right)\Bigg]\\
&= \frac{(t;q)_m(t;q)_n(q^{1-m}/v;q)_m}{(v;q)_m(q;q)_n(q^{-m};q)_m}\left(\frac{v}{q}\right)^m\\
&\qquad\times\left(-\sum_{k=0}^{n-1}\frac{tq^k}{1-tq^k}+\sum_{k=0}^{m-1}\frac{1}{1-vq^k}+\sum_{k=1}^{n}\frac{q^k}{1-q^k}-\sum_{k=1}^{m}\frac{1}{1-q^k}\right)\\
&= \frac{(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left[\sum_{k=1}^{n}\left(\frac{q^k}{1-q^k}-\frac{tq^{k-1}}{1-tq^{k-1}}\right)+\sum_{k=1}^{m}\left(\frac{1}{1-vq^{k-1}}-\frac{1}{1-q^k}\right)\right]\\
&= \frac{(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{n}\frac{q^k(1-t/q)}{(1-tq^{k-1})(1-q^k)}-\sum_{k=1}^{m}\frac{q^k(1-v/q)}{(1-vq^{k-1})(1-q^k)}\right).
\end{aligned}$$

According to (3.3), we get (3.1). □

When we set $v=t$ in (3.1), we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{(t;q)_{n-k}(t;q)_{m+k}(q/t;q)_k}{(q^k;q)_{m+1}(q;q)_{n-k}(q;q)_k(t;q)_k}t^k-\sum_{k=1}^{m}\frac{(t;q)_{m-k}(t;q)_{n+k}(q/t;q)_k}{(q^k;q)_{n+1}(q;q)_{m-k}(q;q)_k(t;q)_k}t^k\\
&\quad=\frac{(1-t/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-tq^{k-1})(1-q^k)}-\sum_{k=1}^{n}\frac{q^k}{(1-tq^{k-1})(1-q^k)}\right),
\end{aligned}\tag{3.4}$$

which is $v=t$ case of the following theorem given by Guo and Zhang in [10].

Theorem 3.2. *[10, Theorem 1.3] For $m,n\geq 0$, we have*

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}t^k-\sum_{k=1}^{m}\frac{(q/t;q)_k(vq^n;q)_k(t;q)_{m-k}(t;q)_n}{(q;q)_k(v;q)_k(q;q)_{m-k}(q^k;q)_{n+1}}t^k\\
&\quad=\frac{(1-t/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-tq^{k-1})(1-q^k)}-\sum_{k=1}^{n}\frac{q^k}{(1-tq^{k-1})(1-q^k)}\right).
\end{aligned}$$

Interchanging m, n in (3.1), we have

$$\sum_{k=1}^{m}\frac{(q/t;q)_k(vq^n;q)_k(t;q)_{m-k}(t;q)_n}{(q;q)_k(v;q)_k(q;q)_{m-k}(q^k;q)_{n+1}}t^k-\sum_{k=1}^{n}\frac{(tq^k;q)_{n-k}(t;q)_{m+k}(q/v;q)_k}{(q^k;q)_{m+1}(vq^{n-k};q)_k(q;q)_{n-k}(q;q)_k}v^k$$
$$=\frac{(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{n}\frac{q^k(1-v/q)}{(1-vq^{k-1})(1-q^k)}-\sum_{k=1}^{m}\frac{q^k(1-t/q)}{(1-tq^{k-1})(1-q^k)}\right). \quad (3.5)$$

Combining (3.1) and (3.5), we get

$$\sum_{k=1}^{n}\frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}t^k-\sum_{k=1}^{m}\frac{(q/t;q)_k(vq^n;q)_k(t;q)_{m-k}(t;q)_n}{(q;q)_k(v;q)_k(q;q)_{m-k}(q^k;q)_{n+1}}t^k$$
$$=\sum_{k=1}^{m}\frac{(tq^k;q)_{m-k}(t;q)_{n+k}(q/v;q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}v^k-\sum_{k=1}^{n}\frac{(tq^k;q)_{n-k}(t;q)_{m+k}(q/v;q)_k}{(q^k;q)_{m+1}(vq^{n-k};q)_k(q;q)_{n-k}(q;q)_k}v^k$$
$$+\frac{(1-v/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-vq^{k-1})(1-q^k)}-\sum_{k=1}^{n}\frac{q^k}{(1-vq^{k-1})(1-q^k)}\right)$$
$$+\frac{(1-t/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-tq^{k-1})(1-q^k)}-\sum_{k=1}^{n}\frac{q^k}{(1-tq^{k-1})(1-q^k)}\right).$$

According to the above identity and Theorem 3.2, we have the following identity:

$$\sum_{k=1}^{n}\frac{(tq^k;q)_{n-k}(t;q)_{m+k}(q/v;q)_k}{(q^k;q)_{m+1}(vq^{n-k};q)_k(q;q)_{n-k}(q;q)_k}v^k-\sum_{k=1}^{m}\frac{(tq^k;q)_{m-k}(t;q)_{n+k}(q/v;q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}v^k$$
$$=\frac{(1-v/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-vq^{k-1})(1-q^k)}-\sum_{k=1}^{n}\frac{q^k}{(1-vq^{k-1})(1-q^k)}\right),$$

which can be obtained by interchanging v and t in Theorem 3.2.

Remark 3.3. *We were not successful to prove [10, Theorem 1.3] with the partial fraction decomposition method. However, we offer the following observation. The righthand-side of it does* not *depend on the parameter v. If this fact could be shown by simple means, then we could argue that the lefthand-side also does not depend on v. Henceforth, we could set $v = t$, and would achieve a proof.*

Next, making some changes on the rational function $F(z)$ in the proof of Theorem 2.1, we can get some new results.

Set

$$F(z)=\frac{(vq;q)_n(tz;q)_{n-l}(tq^{-l}/v;q)_l}{(zv;q)_{n+1}(z-q^{-m})}.$$

Then using the partial fraction decomposition method, we get

$$\sum_{k=0}^{n}\frac{(vq/t;q)_k(tq^{-l}/v;q)_{n-k}}{(q;q)_k(q;q)_{n-k}(1-vq^{k-m})}\left(\frac{t}{v}\right)^k=\frac{(tq^{-m};q)_{n-l}(tq^{-l}/v;q)_l}{(vq^{-m};q)_{n+1}}. \quad (3.6)$$

Setting v and t to be aq^m and aq^{m+1}/b in (3.6), respectively, we get the following result.

Theorem 3.4. *For $n \geq 0$ and $0 \leq l \leq n$, we have*

$$\sum_{k=0}^{n} \frac{(a, b, q^{-n}; q)_k}{(q, aq, bq^{-n+l}; q)_k} (q^{l+1})^k = \frac{(q; q)_n (aq/b; q)_{n-l}}{(aq; q)_n (q/b; q)_{n-l}}. \tag{3.7}$$

Setting $l = 0$ in (3.7), we obtain

$$\sum_{k=0}^{n} \frac{(a, b, q^{-n}; q)_k}{(q, aq, bq^{-n}; q)_k} q^k = \frac{(q, aq/b; q)_n}{(aq, q/b; q)_n},$$

which is $c = aq$ case of the q-Pfaff-Saalschütz summation theorem [9, Appendix II.12]

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ & c, & abq^{1-n}/c \end{matrix} ; q, q \right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.$$

If we set $a = q$ in (3.7), we get

Corollary 3.5. *For $n \geq 0$ and $0 \leq l \leq n$, we have*

$$\sum_{k=0}^{n} \frac{(b, q^{-n}; q)_k}{(q^2, bq^{-n+l}; q)_k} (q^{l+1})^k = \frac{(q; q)_n (q^2/b; q)_{n-l}}{(q^2; q)_n (q/b; q)_{n-l}}. \tag{3.8}$$

When we set $b = q^{n-l+1}$ in (3.8), we get

$$\sum_{k=0}^{n} \frac{(q^{n-l+1}, q^{-n}; q)_k}{(q, q^2; q)_k} (q^{l+1})^k = 0,$$

which is $a = q^{n-l+1}$ and $c = q^2$ case of the q-Chu-Vandermonde identity [9, Appendix II.7]

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix} ; q, \frac{cq^n}{a} \right] = \frac{(c/a; q)_n}{(c; q)_n}.$$

Moreover, we find that if we put a new parameter v in the proofs of Theorem 2.2 and Theorem 2.3, we get generalizations of these two theorems.

First, we give a generalization of Theorem 2.2.

Theorem 3.6. *For $m, n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(vq; q)_n (vq^{m+1}/t; q)_{k-1} (t/v; q)_{n-k}}{(tq^{-m+1}; q)_{m+n-1} (q; q)_{k-1} (q; q)_{n-k} (1 - vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1}$$
$$= \sum_{1 \leq k_m \leq \cdots \leq k_1 \leq n} \frac{q^{k_1 - 1}}{(1 - tq^{k_1 - 1})(1 - vq^{k_1})} \cdots \frac{q^{k_m - 1}}{(1 - tq^{k_m - m})(1 - vq^{k_m})}.$$

Proof. Set

$$F(z) = \frac{(vq; q)_n (tzq^{-m+1}; q)_{m+n-1}}{(vzq; q)_n (tq^{-m}/v; q)_m (tq^{-m+1}; q)_{m+n-1} (z - 1)^{m+1}}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z)=\sum_{k=1}^{n}\frac{b_k}{1-vzq^k}+\sum_{k=1}^{m+1}\frac{c_k}{(1-z)^k}. \tag{3.9}$$

Multiplying both sides of (3.9) by z, and then letting $z\to\infty$, we get

$$\sum_{k=1}^{n}\frac{b_k}{vq^k}=-c_1. \tag{3.10}$$

Now we calculate b_k for $1\le k\le n$.

For $1\le k\le n$, we have

$$\begin{aligned}
b_k&=\lim_{z\to q^{-k}/v}(1-vzq^k)F(z)\\
&=\lim_{z\to q^{-k}/v}\frac{(vq;q)_n(tzq^{-m+1};q)_{m+n-1}}{(vzq;q)_{k-1}(vzq^{k+1};q)_{n-k}(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}(z-1)^{m+1}}\\
&=\frac{(vq;q)_n(tq^{-m-k+1}/v;q)_{m+n-1}}{(q^{1-k};q)_{k-1}(q;q)_{n-k}(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}(q^{-k}/v-1)^{m+1}}\\
&=\frac{(-1)^{k-1}q^{\binom{k+1}{2}+km}v^{m+1}(vq;q)_n(tq^{-m-k+1}/v;q)_{m+n-1}}{(q;q)_{k-1}(q;q)_{n-k}(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}(1-vq^k)^{m+1}}\\
&=\frac{(-1)^{k-1}q^{\binom{k+1}{2}+km}v^{m+1}(vq;q)_n(tq^{-m-k+1}/v;q)_{k-1}(t/v;q)_{n-k}}{(q;q)_{k-1}(q;q)_{n-k}(tq^{-m+1};q)_{m+n-1}(1-vq^k)^{m+1}}\\
&=\frac{(vq;q)_n(vq^{m+1}/t;q)_{k-1}(t/v;q)_{n-k}}{(q;q)_{k-1}(q;q)_{n-k}(tq^{-m+1};q)_{m+n-1}(1-vq^k)^{m+1}}\left(\frac{t}{v}\right)^{k-1}v^{m+1}q^{m+k}.
\end{aligned}$$

For c_1, we have

$$\begin{aligned}
c_1&=\frac{(-1)^m}{m!}\lim_{z\to1}D_z^{(m)}\left\{(1-z)^{m+1}F(z)\right\}\\
&=\frac{(-1)^m}{m!}\lim_{z\to1}D_z^{(m)}\left\{\frac{(-1)^{m+1}(vq;q)_n(tzq^{-m+1};q)_{m+n-1}}{(vzq;q)_n(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}}\right\}\\
&=-\frac{(vq;q)_n}{(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}m!}\lim_{z\to1}D_z^{(m)}\left\{\frac{(tzq^{-m+1};q)_{m+n-1}}{(vzq;q)_n}\right\}\\
&=-\frac{(vq;q)_n}{(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}}[(z-1)^m]\frac{(tzq^{-m+1};q)_{m+n-1}}{(vzq;q)_n}\\
&=-\frac{(vq;q)_n}{(tq^{-m}/v;q)_m(tq^{-m+1};q)_{m+n-1}}[w^m]\frac{(t(w+1)q^{-m+1};q)_{m+n-1}}{(v(w+1)q;q)_n}\\
&=-\frac{1}{(tq^{-m}/v;q)_m}[w^m]\frac{\prod_{k=1}^{m+n-1}\left(1-\frac{twq^{-m+k}}{1-tq^{-m+k}}\right)}{\prod_{k=1}^{n}\left(1-\frac{vwq^k}{1-vq^k}\right)}\\
&=-\frac{1}{(tq^{-m}/v;q)_m}\sum_{1\le k_m\le\cdots\le k_1\le n}\left(\frac{vq^{k_1}}{1-vq^{k_1}}-\frac{tq^{k_1-1}}{1-tq^{k_1-1}}\right)\cdots\left(\frac{vq^{k_m}}{1-vq^{k_m}}-\frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right)
\end{aligned}$$

$$= -\frac{1}{(tq^{-m}/v;q)_m} \sum_{1\le k_m\le\cdots\le k_1\le n} \frac{vq^{k_1}(1-tq^{-1}/v)}{(1-tq^{k_1-1})(1-vq^{k_1})} \cdots \frac{vq^{k_m}(1-tq^{-m}/v)}{(1-tq^{k_m-m})(1-vq^{k_m})}$$

$$= -v^m \sum_{1\le k_m\le\cdots\le k_1\le n} \frac{q^{k_1}}{(1-tq^{k_1-1})(1-vq^{k_1})} \cdots \frac{q^{k_m}}{(1-tq^{k_m-m})(1-vq^{k_m})}.$$

We read off the coefficient of w^m as we did in the proof of Theorem 2.2.

According to (3.10), we get Theorem 3.6. □

Setting $v = 1$ in Theorem 3.6, we get Theorem 2.2.

Setting $v = 0$ in Theorem 3.6, we have

$$\sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} (-1)^k q^{\binom{k+1}{2}-nk} = 0,$$

which is a special case of the q-binomial theorem [9, Appendix II.4]

$${}_1\phi_0 \left[\begin{matrix} q^{-n} \\ - \end{matrix} ; q, z \right] = (zq^{-n};q)_n.$$

Setting $v = q$ in Theorem 3.6, we have the following result.

Corollary 3.7. *For $m, n \ge 1$, we have*

$$\sum_{k=1}^{n} \frac{(q^2;q)_n (q^{m+2}/t;q)_{k-1} (t/q;q)_{n-k}}{(tq^{-m+1};q)_{m+n-1} (q;q)_{k-1} (q;q)_{n-k} (1-q^{k+1})^{m+1}} \left(\frac{t}{q}\right)^{k-1}$$
$$= \sum_{1\le k_m\le\cdots\le k_1\le n} \frac{q^{k_1-1}}{(1-tq^{k_1-1})(1-q^{k_1+1})} \cdots \frac{q^{k_m-1}}{(1-tq^{k_m-m})(1-q^{k_m+1})}.$$

Letting $n \to \infty$ in Theorem 3.6, we have

Corollary 3.8. *For $m \ge 1$, we have*

$$\sum_{k=1}^{\infty} \frac{(vq^{m+1}/t;q)_{k-1}}{(tq^{-m+1};q)_{m-1} (q;q)_{k-1} (1-vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1}$$
$$= \frac{(q;q)_\infty (t;q)_\infty}{(vq;q)_\infty (t/v;q)_\infty} \sum_{1\le k_m\le\cdots\le k_1<\infty} \frac{q^{k_1-1}}{(1-tq^{k_1-1})(1-vq^{k_1})} \cdots \frac{q^{k_m-1}}{(1-tq^{k_m-m})(1-vq^{k_m})}.$$

Setting $m = 1$ in Theorem 3.6, we get the following generalization of [10, Corollary 3.3].

Corollary 3.9. *For $n \ge 1$, we have*

$$\sum_{k=1}^{n} \frac{(vq;q)_n (vq^2/t;q)_{k-1} (t/v;q)_{n-k}}{(t;q)_n (q;q)_{k-1} (q;q)_{n-k} (1-vq^k)^2} \left(\frac{t}{v}\right)^{k-1} = \sum_{k=1}^{n} \frac{q^{k-1}}{(1-tq^{k-1})(1-vq^k)}.$$

Now we give the following theorem which is a generalization of Theorem 2.3.

Theorem 3.10. *For $m, n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(vq;q)_n (vq^{m+1}/t;q)_{k-1} (tq^{-1}/v;q)_{n-k}}{(tq^{-m+1};q)_{m+n-2} (q;q)_{k-1} (q;q)_{n-k} (1-vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1}$$
$$= \sum_{k_1=1}^{n} \frac{q^{k_1-1}}{(1-tq^{k_1-2})(1-vq^{k_1})} \cdots \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}-1}}{(1-tq^{k_{m-1}-m})(1-vq^{k_{m-1}})}$$
$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m-1}}{1-vq^{k_m}} - \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m-1}}{v-tvq^{k_m-m}} \right).$$

We perform partial fraction decomposition on

$$F(z) = \frac{(vq;q)_n (tzq^{-m+1};q)_{m+n-2}}{(vzq;q)_n (tq^{-m}/v;q)_{m-1} (tq^{-m+1};q)_{m+n-2} (z-1)^{m+1}}.$$

Similarly to the proof of Theorem 2.3, we can prove the above theorem. Here we omit the proof.

When we set $v = 1$ in Theorem 3.10, we get Theorem 2.3.

Setting $v = q$ in Theorem 3.10, we have

Corollary 3.11. *For $m, n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(q^2;q)_n (q^{m+2}/t;q)_{k-1} (tq^{-2};q)_{n-k}}{(tq^{-m+1};q)_{m+n-2} (q;q)_{k-1} (q;q)_{n-k} (1-q^{k+1})^{m+1}} \left(\frac{t}{q}\right)^{k-1}$$
$$= \sum_{k_1=1}^{n} \frac{q^{k_1-1}}{(1-tq^{k_1-2})(1-q^{k_1+1})} \cdots \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}-1}}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}+1})}$$
$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m-1}}{1-q^{k_m+1}} - \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m-2}}{1-tq^{k_m-m}} \right).$$

Letting $n \to \infty$ in Theorem 3.10, we obtain

Corollary 3.12. *For $m \geq 1$, we have*

$$\sum_{k=1}^{\infty} \frac{(vq^{m+1}/t;q)_{k-1}}{(tq^{-m+1};q)_{m-1} (q;q)_{k-1} (1-vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1} = \frac{(q;q)_\infty (t;q)_\infty}{(vq;q)_\infty (tq^{-1}/v;q)_\infty}$$
$$\sum_{k_1=1}^{\infty} \frac{q^{k_1-1}}{(1-tq^{k_1-2})(1-vq^{k_1})} \cdots \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}-1}}{(1-tq^{k_{m-1}-m})(1-vq^{k_{m-1}})}$$
$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m-1}}{1-vq^{k_m}} - \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m-1}}{v-tvq^{k_m-m}} \right).$$

Setting $m = 1$ in Theorem 3.10, we have the following generalization of the $l = 1$ case of [10, Corollary 3.2].

Corollary 3.13. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{(vq;q)_n (vq^2/t;q)_{k-1} (tq^{-1}/v;q)_{n-k}}{(t;q)_{n-1}(q;q)_{k-1}(q;q)_{n-k}(1-vq^k)^2} \left(\frac{t}{v}\right)^{k-1} = \sum_{k=1}^{n} \frac{q^{k-1}}{1-vq^k} - \sum_{k=1}^{n-1} \frac{tq^{k-2}}{v-tvq^{k-1}}.$$

4. Two identities of Fang

We notice that by using partial fraction decomposition technique, we can also give new proofs of two identities found by Fang in [8].

Theorem 4.1. *[8, Corollary 3.4] We have*

$$\sum_{j=0}^{M} \begin{bmatrix} M \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}+2j} \frac{1}{1-zq^j} = \frac{(q;q)_M}{(z;q)_{M+1}} \sum_{j=0}^{M} (z;q)_j q^j.$$

Proof. Set

$$F(z) = \frac{(q;q)_M}{(z;q)_{M+1}} \sum_{j=0}^{M} (z;q)_j q^j.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=0}^{M} \frac{b_k}{1-zq^k}. \tag{4.1}$$

For $0 \leq k \leq M$, we have

$$\begin{aligned} b_k &= \lim_{z \to q^{-k}} (1-zq^k) F(z) \\ &= \lim_{z \to q^{-k}} \frac{(q;q)_M}{(z;q)_k (zq^{k+1};q)_{M-k}} \sum_{j=0}^{M} (z;q)_j q^j \\ &= \frac{(q;q)_M}{(q^{-k};q)_k (q;q)_{M-k}} \sum_{j=0}^{M} (q^{-k};q)_j q^j \\ &= (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} M \\ k \end{bmatrix} \sum_{j=0}^{k} (q^{-k};q)_j q^j \\ &= \begin{bmatrix} M \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}+2k}. \end{aligned}$$

The last equation of this chain of equalities follows from the q-Chu-Vandermonde identity [9, Appendix II.6]

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix} ; q, q \right] = \frac{(c/a;q)_n}{(c;q)_n} a^n \tag{4.2}$$

by setting $a = q$ and $c = 0$.

According to (4.1), we get Theorem 4.1. □

Theorem 4.2. *[8, Corollary 4.1] We have*

$$\sum_{j=0}^{M} \frac{q^j}{(z;q)_{j+1}} = \sum_{j=0}^{M} \frac{(-1)^j q^{\binom{j}{2}+2j}}{(1-zq^j)(q;q)_j(q;q)_{M-j}}.$$

Proof. Set

$$F(z) = \sum_{j=0}^{M} \frac{q^j}{(z;q)_{j+1}}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=0}^{M} \frac{b_k}{1-zq^k}. \tag{4.3}$$

For $0 \le k \le M$, we have

$$\begin{aligned}
b_k &= \lim_{z\to q^{-k}} (1-zq^k)F(z) \\
&= \lim_{z\to q^{-k}} (1-zq^k) \sum_{j=0}^{M} \frac{q^j}{(z;q)_{j+1}} \\
&= \lim_{z\to q^{-k}} (1-zq^k) \sum_{j=k}^{M} \frac{q^j}{(z;q)_{j+1}} \\
&= \lim_{z\to q^{-k}} \sum_{j=k}^{M} \frac{q^j}{(z;q)_k (zq^{k+1};q)_{j-k}} \\
&= \sum_{j=k}^{M} \frac{q^j}{(q^{-k};q)_k (q;q)_{j-k}} \\
&= \frac{(-1)^k q^{\binom{k}{2}+2k}}{(q;q)_k} \sum_{j=0}^{M-k} \frac{q^j}{(q;q)_j} \\
&= \frac{(-1)^k q^{\binom{k}{2}+2k}}{(q;q)_k (q;q)_{M-k}}.
\end{aligned}$$

The last equation follows from the q-Chu-Vandermonde identity (4.2) by setting $a = 0$ and $c = q^{-n}$.

According to (4.3), we complete the proof of Theorem 4.2. □

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