A PROOF OF MELHAM'S CONJECTURE

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Abstract. In this paper, we consider Melham's conjecture involving Fibonacci and Lucas numbers. After rewriting it in terms of Fibonomial coefficients, we give a solution of the conjecture by evaluating a certain q-sum using contour integration.

1. Introduction

The Fibonacci numbers are defined for $n > 0$ by

$$F_{n+1} = F_n + F_{n-1}$$

where $F_0 = 0$ and $F_1 = 1$.

The Lucas numbers are defined for $n > 0$ by

$$L_{n+1} = L_n + L_{n-1}$$

where $L_0 = 2$ and $L_1 = 1$.

The Binet forms of the Fibonacci and Lucas sequences are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n$$

where α, β are $(1 \pm \sqrt{5}\,)/2$.

Recently as an interesting generalization of the binomial coefficients, the Fibonomial coefficients have been taken interest of several authors. For their properties, we refer to [1, 6, 8, 9, 11, 14, 15, 17, 19, 21, 22]. The Fibonomial coefficient is defined by the relation for $n \geq m \geq 1$

$$\left\{ {n \atop m} \right\} = \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_{n-m})\,(F_1 F_2 \dots F_m)} = \frac{F_n!}{F_{n-m}! F_m!}$$

with $\left\{ {n \atop n} \right\} = \left\{ {n \atop 0} \right\} = 1$ where F_n is the nth Fibonacci number and $F_n! = F_1 F_2 \dots F_n$ is nth Fibonacci factorial.

These coefficients satisfy the relation:

$$\left\{ {n \atop m} \right\} = F_{m+1} \left\{ {n-1 \atop m} \right\} + F_{n-m-1} \left\{ {n-1 \atop m-1} \right\}. \tag{1.1}$$

In [8], Hoggatt considered Fibonomial coefficients with indices in arithmetic progressions. For example, he defined the following generalization by taking F_{kn} instead of F_n for a fixed positive integer k:

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$$\left\{ {n \atop m} \right\}_k = \frac{F_k F_{2k} \dots F_{kn}}{\left(F_k F_{2k} \dots F_{k(n-m)}\right)\left(F_k F_{2k} \dots F_{km}\right)}.$$

The Fibonomials appear in several places in the literature; we give two examples: The $n \times n$ right-adjusted Pascal matrix P_n whose (i,j) entry is given by

$$(P_n)_{ij} = \binom{j-1}{j+i-n-1}.$$

Carlitz [2] firstly showed that the Fibonomial coefficients appear in the auxiliary polynomial of the right-adjusted Pascal matrix P_n as the coefficients of it. Since some relationships between the generalized Pascal matrix and the Fibonomial coefficients have been constructed, the Fibonomial coefficients have been studied by some authors [2, 3, 12, 13, 20].

Secondly, in [5, 7, 8, 10], one can find that the nth powers of Fibonacci numbers satisfy the following auxiliary polynomial

$$C_n(x) = \sum_{i=0}^{n} (-1)^{i(i+1)/2} \left\{ {n \atop i} \right\} x^{n-i} \tag{1.2}$$

where $\left\{ {n \atop i} \right\}$ is defined as before.

Torreto and Fuchs [21] considered the general second order recurrence relation

$$y_{n+2} = g y_{n+1} - h y_n, \qquad h \neq 0. \tag{1.3}$$

Let α and β be the roots of the auxiliary polynomial $f(x) = x^2 - gx + h$ of (1.3). Let U_n and V_n be two solutions of (1.3) defined by

$$U_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } \alpha \neq \beta \\ n\alpha^{n-1} & \text{if } \alpha = \beta \end{cases} \qquad \text{and} \qquad V_n = \alpha^n + \beta^n.$$

In [10], Jarden showed that the kth order recurrence relation

$$\sum_{j=0}^{k} (-1)^j \left\{ {k \atop j} \right\}_U h^{\frac{j(i+1)}{2}} z_{n+k-j} = 0$$

holds for the product z_n of the nth terms of $k-1$ sequences satisfying (1.3), where

$$\left\{ {k \atop j} \right\}_U = \frac{U_k U_{k-1} \dots U_{k-j+1}}{U_1 U_2 \dots U_j}, \qquad \left\{ {k \atop 0} \right\}_U = 1.$$

In [21], the authors established some identities involving the $\left\{ {k \atop j} \right\}_U$, one of which is the formula:

$$\sum_{j=0}^{k} (-1)^j \left\{ {k \atop j} \right\}_U h^{\frac{j(i+1)}{2}} U_{a_1+k-j} U_{a_2+k-j} \dots U_{a_k+k-j} y_{n+k-j}$$
$$= U_1 U_2 \dots U_k y_{n+a_1+\dots+a_k+k(k+1)/2},$$

where y_n and U_n satisfying (1.3) and n and the a's are any integers.

Melham [16] derived the families of identities between sums of powers of the Fibonacci and Lucas numbers. In his work, while deriving these identities, he conjectured a complex identity between the Fibonacci and Lucas numbers. Now we recall this conjecture:

(a) Let $k, m, n \in \mathbb{Z}$ with $m > 0$, show that

$$\sum_{j=0}^{m-1} \frac{F_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m+3)}{2}} \frac{F_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = F_{(m+1)(n+\frac{m}{2})}. \quad (1.4)$$

(b) The Lucas counterpart of (a) is given by

$$\sum_{j=0}^{m-1} \frac{L_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m+3)}{2}} \frac{L_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = \begin{cases} 5^{\frac{m+1}{2}} F_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} L_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is even,} \end{cases}$$

where $(F_n)_{(m)}$ is the "falling" factorial, which begins at F_n for $n \neq 0$, and is the product of m Fibonacci numbers excluding F_0. For example $(F_6)_{(5)} = F_6F_5F_4F_3F_2$ and $(F_3)_{(5)} = F_3F_2F_1F_{-1}F_{-2}$. For $m > 0$, define $(F_0)_{(m)} = F_{-1}F_{-2} \ldots F_{-m}$ and $(F_0)_{(0)} = 1$.

In this paper, first we rearrange the conjecture of Melham by using Fibonomial coefficients instead of the "falling Fibonacci factorial". After this, we give a solution of the conjecture by translating it into a q-expression; we are left with the evaluation of a certain sum. This is achieved using contour integration.

2. The conjecture in terms of Fibonomials

In this section, before solving the conjecture, we will need to rewrite it via the Fibonomials. One can obtain the following version of the conjecture (a) by considering the definitions of the falling Fibonacci factorial and the Fibonomials coefficients:

Conjecture 1. *For any integers m, n and k,*

$$\sum_{j=0}^{m-1} (-1)^{\frac{j(j+3)}{2}} \left\{ {(m+1)k+m \atop j} \right\} \left\{ {(m+1)k+m-j-1 \atop m-j-1} \right\} F_{n+k+m-j}^{m+1} + (-1)^{\frac{m(m+3)}{2}} F_{n-mk}^{m+1} = \left(\prod_{j=1}^{m} F_{(m+1)k+j} \right) F_{(m+1)(n+\frac{m}{2})}.$$

Proof. By taking $q = \beta/\alpha$, the claimed equality is reduced to the following form:

$$(1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k+m-1 \\ m-1 \end{bmatrix}_q \times \sum_{j=0}^{m-1} (-1)^j q^{j(j+1)/2} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \frac{(1-q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}}$$
$$= \left(1 - q^{\frac{(m+1)(2n+m)}{2}}\right) \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^m q^{\frac{m(m+1)(2k+1)}{2}} (1-q^{n-mk})^{m+1} \quad (2.1)$$

where $\begin{bmatrix} m \\ j \end{bmatrix}_q$ stands for the Gaussian q-binomial coefficient:

$$\begin{bmatrix} m \\ j \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}}$$

with

$$(z;q)_n := (1-z)(1-zq)\dots(1-zq^{n-1}).$$

Define

$$S := \sum_{j=1}^{m-1} (-1)^j q^{j(j+1)/2} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \frac{(1-q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}}.$$

By contour integration (for similar examples, see [18]),

$$I_R = \frac{1}{2\pi i} \oint \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}}\, dz.$$

The integration is over a large circle with radius R. We evaluate the integral by residues. The poles at $q^{-1}, \dots, q^{-(m-1)}$ lead to our sum, but there are other poles: at $z=0$, at $z=1$, and at $z = q^{-(m+1)k-m}$. Then we get

$$\begin{aligned} I_R = -S &+ \operatorname{Res}_{z=0} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}} \\ &+ \operatorname{Res}_{z=1} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}} \\ &+ \operatorname{Res}_{z=q^{-(m+1)k-m}} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}}. \end{aligned}$$

Note that as $|z|$ gets large,

$$\begin{aligned} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}} &\sim \frac{1}{z} \frac{(q;q)_{m-1}}{q^{\binom{m}{2}}} \frac{q^{(n+k+m)(m+1)}}{q^{(m+1)k+m}} \\ &= \frac{1}{z} (q;q)_{m-1} q^{n(m+1)+\frac{m(m+1)}{2}}. \end{aligned}$$

Consequently, as $R \to \infty$,

$$I_R \to (q;q)_{m-1} q^{n(m+1)+\frac{m(m+1)}{2}}.$$

Thus

$$\begin{aligned} (q;q)_{m-1} q^{n(m+1)+\frac{m(m+1)}{2}} &= -S + (q;q)_{m-1} - \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}} \\ &+ [(z-q^{-(m+1)k-m})^{-1}] \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}}. \end{aligned}$$

So we obtain

$$
\begin{aligned}
S &= -(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}} + (q;q)_{m-1} - \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}} \\
&\quad - \frac{(q;q)_{m-1}}{(q^{-(m+1)k-m};q)_m}(1-q^{n-mk})^{m+1} \\
&= -(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}} + (q;q)_{m-1} - \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}} \\
&\quad - (-1)^m \frac{(q;q)_{m-1}(q;q)_{(m+1)k}q^{m(m+1)k+\frac{m(m+1)}{2}}}{(q;q)_{(m+1)k+m}}(1-q^{n-mk})^{m+1}.
\end{aligned}
$$

Now we must prove that

$$
\begin{aligned}
&(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}} \\
&- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}} \\
&+ (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1} \\
&- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}} \\
&- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q \times \\
&\qquad \times (-1)^m \frac{(q;q)_{m-1}(q;q)_{(m+1)k}q^{m(m+1)k+\frac{m(m+1)}{2}}}{(q;q)_{(m+1)k+m}}(1-q^{n-mk})^{m+1} \\
&= \left(1-q^{\frac{(m+1)(2n+m)}{2}}\right)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^m q^{\frac{m(m+1)(2k+1)}{2}}(1-q^{n-mk})^{m+1}.
\end{aligned}
$$

We gradually simplify the equation that must be proved:

$$
\begin{aligned}
&- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}} \\
&+ (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1} \\
&- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q \times \\
&\qquad \times (-1)^m \frac{(q;q)_{m-1}(q;q)_{(m+1)k}q^{m(m+1)k+\frac{m(m+1)}{2}}}{(q;q)_{(m+1)k+m}}(1-q^{n-mk})^{m+1} \\
&= (1-q^{\frac{(m+1)(2n+m)}{2}})\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^m q^{\frac{m(m+1)(2k+1)}{2}}(1-q^{n-mk})^{m+1},
\end{aligned}
$$

or

$$\begin{aligned}
&-(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}}\\
&+(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1}\\
&=(1-q^{\frac{(m+1)(2n+m)}{2}})\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}},
\end{aligned}$$

or

$$\begin{aligned}
(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1}\Big(1-q^{n(m+1)+\frac{m(m+1)}{2}}\Big)\\
=\Big(1-q^{\frac{(m+1)(2n+m)}{2}}\Big)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}.
\end{aligned}$$

Consequently we are left to prove that

$$(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q (q;q)_{m-1}=\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}.$$

Since this is true, the proof is complete. ∎

Note that this proof establishes (2.1) for all values of q, not just for $q=\beta/\alpha$.

Similarly the Lucas counterpart of the conjecture is rewritten in terms of the Fibonomials as follows: for odd m,

$$\begin{aligned}
&\sum_{j=0}^{m-1}(-1)^{\frac{j(j+3)}{2}}\left\{\begin{matrix}(m+1)k+m\\ j\end{matrix}\right\}\left\{\begin{matrix}(m+1)k+m-j-1\\ m-j-1\end{matrix}\right\}L_{n+k+m-j}^{m+1}\\
&\qquad+(-1)^{\frac{m(m+3)}{2}}L_{n-mk}^{m+1}=5^{\frac{m+1}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+j}\right)F_{(m+1)(n+\frac{m}{2})},
\end{aligned}$$

and for even m,

$$\begin{aligned}
&\sum_{j=0}^{m-1}(-1)^{\frac{j(j+3)}{2}}\left\{\begin{matrix}(m+1)k+m\\ j\end{matrix}\right\}\left\{\begin{matrix}(m+1)k+m-j-1\\ m-j-1\end{matrix}\right\}L_{n+k+m-j}^{m+1}\\
&\qquad+(-1)^{\frac{m(m+3)}{2}}L_{n-mk}^{m+1}=5^{\frac{m}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+j}\right)L_{(m+1)(n+\frac{m}{2})}.
\end{aligned}$$

By taking $q=\beta/\alpha$, the above equalities are translated to the following forms:

$$\begin{aligned}
&(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q\times\\
&\sum_{j=0}^{m-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}m-1\\ j\end{bmatrix}_q\frac{(1+q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}} \qquad (2.2)\\
&=\Big(1-q^{\frac{(m+1)(2n+m)}{2}}\Big)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}-(-1)^m q^{\frac{m(m+1)(2k+1)}{2}}(1+q^{n-mk})^{m+1}
\end{aligned}$$

and

$$
\begin{aligned}
&(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\ m-1\end{bmatrix}_q \times\\
&\sum_{j=0}^{m-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}m-1\\ j\end{bmatrix}_q \frac{(1+q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}} \qquad (2.3)\\
&=\Big(1+q^{\frac{(m+1)(2n+m)}{2}}\Big)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}-(-1)^m q^{\frac{m(m+1)(2k+1)}{2}}(1+q^{n-mk})^{m+1}
\end{aligned}
$$

respectively. Note that (2.2) and (2.3) are the same; it is no more necessary to distinguish the parity of m.

The proof of the equality (2.2) (and (2.3)) can be done similarly to (2.1). Again, it holds for general q.

References

[1] J. E. Andersen and C. Berg, Quantum Hilbert matrices and orthogonal polynomials, J. Comput. Appl. Math, in press.

[2] L. Carlitz, The characteristic polynomial of a certain matrix of binomial coefficients, Fibonacci Quart. 3 (1965), 81–89.

[3] C. Cooper and R. Kennedy, Proof of a result by Jarden by generalizing a proof by Carlitz, Fibonacci Quart. 33 (4) (1995), 304–310.

[4] L. A. G. Dresel, Transformations of Fibonacci-Lucas identities, In Applications of Fibonacci Numbers, 5 (1993), 169–184.

[5] P. Duvall and T. Vaughan, Pell polynomials and a conjecture of Mahon and Horadam, Fibonacci Quart. 26 (4) (1988), 344–353.

[6] H. W. Gould, The bracket function and Fountené–Ward generalized binomial coefficients with application to Fibonomial coefficients, Fibonacci Quart. 7 (1969), 23–40.

[7] A. P. Hillman and V. E. Hoggatt, The characteristic polynomial of the generalized shift matrix, Fibonacci Quart. 3 (2) (1965), 91–94.

[8] V. E. Hoggatt Jr., Fibonacci numbers and generalized binomial coefficients, Fibonacci Quart. 5 (1967), 383–400.

[9] A. F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, Duke Math. J. 32 (1965), 437–446.

[10] D. Jarden, Recurring sequences, Riveon Lematematika, Jerusalem, Israel, 1958, 42–45.

[11] D. Jarden, T. Motzkin, The product of sequences with a common linear recursion formula of order 2, Riveon Lematematika 3, (1949) 25–27, 38.

[12] E. Kilic, The generalized Fibonomial matrix, European J. Comb., in press.

[13] E. Kilic, G. N. Stanica and P. Stanica, Spectral Properties of Some Combinatorial Matrices, 13th International Conference on Fibonacci Numbers and Their Applications, 2008.

[14] D. A. Lind, A determinant involving generalized Binomial coefficients, Fibonacci Quart. 9 (2) (1971) 113–119, 162.

[15] C. T. Long, Discovering Fibonacci identities, Fibonacci Quarterly, 24 (2) (1986), 160–167.

[16] R. S. Melham, Families of identities involving sums of powers of the Fibonacci and Lucas numbers, Fibonacci Quarterly, 37 (4) (1999), 315–319.

[17] H. Prodinger, On a question of Cooper and Kennedy. H. Prodinger, Fibonacci Quarterly 35 (1997) 135–136.

[18] H. Prodinger, Some applications of the q-Rice formula, Random Structures and Algorithms 19 (3-4) (2001), 552–557.

[19] J. Seibert and P. Trojovsky, On some identities for the Fibonomial coefficients, Math. Slovaca 55 (2005), 9–19.

[20] P. Stanica, Netted matrices, Int. J. Math. Math. Sci. 39 (2003), 2507–2518.

[21] R. F. Torretto and J. A. Fuchs, Generalized Binomial coefficients, Fibonacci Quart. 2 (1964), 296–302.

[22] P. Trojovsky, On some identities for the Fibonomial coefficients via generating function. Discrete Appl. Math. 155 (15) (2007), 2017–2024.

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