ON RUEHR'S IDENTITIES

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Abstract. We apply completely elementary tools to achieve recursion formulas for four polynomials with binomial coefficients. In particular, we obtain simple new proofs for Ruehr's combinatorial identities. Moreover, we use our formulas to find identities and inequalities for trigonometric polynomials.

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1. Introduction and main result

We define the following four sums:

$$A_n = \sum_{j=0}^{n} 3^j \binom{3n-j}{2n}, \qquad B_n = \sum_{j=0}^{n} 2^j \binom{3n+1}{n-j},$$

$$C_n = \sum_{j=0}^{2n} (-3)^j \binom{3n-j}{n}, \qquad D_n = \sum_{j=0}^{2n} (-4)^j \binom{3n+1}{n+1+j}.$$

The study of two integral equations led Ruehr [6] to the identities

$$A_n = C_n \quad \text{and} \quad B_n = D_n \quad (n = 0, 1, 2, \ldots). \tag{1}$$

We remark that in [6] A_n is erroneously given with 4^j instead of 3^j. The corrected version is due to Meehan et al. [8].

In a recently published paper, Meehan et al. [8] present new computer-generated proofs for (1) by using the Wilf-Zeilberger method. Moreover, they offer an interesting combinatorial proof for

$$A_n = B_n \quad (n = 0, 1, 2, \ldots).$$

In particular, the authors show that A_n, B_n, C_n, and D_n satisfy the recursion formula

$$X_0 = 1, \quad X_{n+1} = \frac{27}{4} X_n - \frac{3}{4(n+1)} \binom{3n+1}{n} \quad (n = 0, 1, 2, \ldots). \tag{2}$$

In this note, we establish the recursions in the simplest possible way, by only using the recursion

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

of Pascal's triangle and elementary rearrangements. Actually, we prove a bit more: in the next section we demonstrate that the four polynomials

$$A_n(x) = \sum_{j=0}^{n} \binom{3n-j}{2n} x^j, \qquad B_n(x) = \sum_{j=0}^{n} \binom{3n+1}{n-j} x^j,$$

$$C_n(x) = \sum_{j=0}^{2n} \binom{3n-j}{n} x^j, \qquad D_n(x) = \sum_{j=0}^{2n} \binom{3n+1}{n+1+j} x^j$$

satisfy the following recursion formulas.

Theorem. *For all $n \geq 0$ we have*

$$A_{n+1}(x) = \frac{x^3}{(x-1)^2} A_n(x) + \frac{(4n+2)x^2 - (15n+10)x + 9n + 6}{2(n+1)(x-1)^2} \binom{3n+1}{n},$$

$$B_{n+1}(x) = \frac{(x+1)^3}{x^2} B_n(x) + \frac{(4n+2)x^2 - (7n+6)x - 2n - 2}{2(n+1)x^2} \binom{3n+1}{n},$$

$$C_{n+1}(x) = \frac{x^3}{x-1}C_n(x) + \frac{(2n+2)x^2+(3n+2)x-9n-6}{2(n+1)(x-1)}\binom{3n+1}{n},$$
$$D_{n+1}(x) = \frac{(x+1)^3}{x}D_n(x) + \frac{(2n+2)x^2+(7n+6)x-4n-2}{2(n+1)x}\binom{3n+1}{n}.$$

Remark 1. From the recursions we obtain the identities

$$A_n(x+1) = B_n(x) \quad \text{and} \quad C_n(x+1) = D_n(x). \tag{3}$$

The fact that $A_n(x+1) = B_n(x)$ can be seen directly:

$$\begin{aligned} A_n(x+1) &= \sum_{j=0}^{n}(x+1)^j\binom{3n-j}{2n} = \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{j}{k}x^k\binom{3n-j}{2n} \\ &= \sum_{k=0}^{n}x^k\sum_{j=k}^{n}\binom{j}{k}\binom{3n-j}{2n} = \sum_{k=0}^{n}x^k\binom{3n+1}{2n+1+k} \\ &= \sum_{k=0}^{n}x^k\binom{3n+1}{n-k} = B_n(x). \end{aligned}$$

The identity that was used here is a variant of the Vandermonde convolution [5].

The direct proof that $C_n(x+1) = D_n(x)$ is similar.

Remark 2. We have $A_n(3) = A_n$, $B_n(2) = B_n$, $C_n(-3) = C_n$, and $D_n(-4) = D_n$. From the Theorem we conclude that A_n, B_n, C_n and D_n satisfy the recursion formula (2). In particular, we obtain $A_n = B_n = C_n = D_n$ for $n \geq 0$.

In the next section, we prove our theorem and in Section 3 we show that (3) can be applied to obtain identities and inequalities for trigonometric polynomials.

2. Proof

Let us start with the simpler ones.

$$\begin{aligned} B_{n+1}(x) &= \sum_{j=0}^{n+1}x^j\binom{3n+4}{n+1-j} \\ &= \sum_{j=0}^{n+1}x^j\left[\binom{3n+1}{n+1-j} + 3\binom{3n+1}{n-j} + 3\binom{3n+1}{n-1-j} + \binom{3n+1}{n-2-j}\right] \\ &= x\sum_{j=-1}^{n}x^j\binom{3n+1}{n-j} + 3\sum_{j=0}^{n}x^j\binom{3n+1}{n-j} \\ &\quad + \frac{3}{x}\sum_{j=1}^{n}x^j\binom{3n+1}{n-j} + \frac{1}{x^2}\sum_{j=2}^{n}x^j\binom{3n+1}{n-j} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(x+1)^3}{x^2}\sum_{j=0}^{n} x^j\binom{3n+1}{n-j}\\
&\qquad + \binom{3n+1}{n+1} - \frac{3}{x}\binom{3n+1}{n} - \frac{1}{x^2}\binom{3n+1}{n} - \frac{1}{x}\binom{3n+1}{n-1}\\
&= \frac{(x+1)^3}{x^2}B_n(x) + \frac{(4n+2)x^2-(7n+6)x-2n-2}{2(n+1)x^2}\binom{3n+1}{n}.
\end{aligned}
$$

$$
\begin{aligned}
D_{n+1}(x) &= \sum_{j=0}^{2n+2} x^j\binom{3n+4}{n+2+j}\\
&= \sum_{j=0}^{2n+2} x^j\left[\binom{3n+1}{n+2+j} + 3\binom{3n+1}{n+1+j} + 3\binom{3n+1}{n+j} + \binom{3n+1}{n-1+j}\right]\\
&= \sum_{j=0}^{2n-1} x^j\binom{3n+1}{n+2+j} + 3\sum_{j=0}^{2n} x^j\binom{3n+1}{n+1+j}\\
&\qquad + 3\sum_{j=0}^{2n+1} x^j\binom{3n+1}{n+j} + \sum_{j=0}^{2n+2} x^j\binom{3n+1}{n-1+j}\\
&= \frac{1}{x}\sum_{j=1}^{2n} x^j\binom{3n+1}{n+1+j} + 3\sum_{j=0}^{2n} x^j\binom{3n+1}{n+1+j}\\
&\qquad + 3x\sum_{j=-1}^{2n} x^j\binom{3n+1}{n+1+j} + x^2\sum_{j=-2}^{2n} x^j\binom{3n+1}{n+1+j}\\
&= \frac{(x+1)^3}{x}D_n(x) - \frac{1}{x}\binom{3n+1}{n+1} + 3\binom{3n+1}{n} + \binom{3n+1}{n-1} + x\binom{3n+1}{n}\\
&= \frac{(x+1)^3}{x}D_n(x) + \frac{(2n+2)x^2+(7n+6)x-4n-2}{2(n+1)x}\binom{3n+1}{n}.
\end{aligned}
$$

Now we move to the two remaining sums.

$$
\begin{aligned}
A_{n+1}(x) &= \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+2+j}{j}\\
&= \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+1+j}{j} + \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+1+j}{j-1}\\
&= \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+1+j}{j} + \sum_{j=0}^{n+1} x^{n-j}\binom{2n+2+j}{j} - \frac{1}{x}\binom{3n+3}{n+1}\\
&= \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+1+j}{j} + \frac{1}{x}A_{n+1}(x) - \frac{1}{x}\binom{3n+3}{n+1}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\Big(1-\frac{1}{x}\Big)A_{n+1}(x)+\frac{1}{x}\binom{3n+3}{n+1} &= \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+1+j}{j} \\
&= \sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+j}{j}+\sum_{j=0}^{n+1} x^{n+1-j}\binom{2n+j}{j-1} \\
&= xA_n(x)+\binom{3n+1}{n+1}+\sum_{j=0}^{n+1} x^{n-j}\binom{2n+1+j}{j}-\frac{1}{x}\binom{3n+2}{n+1} \\
&= xA_n(x)+\binom{3n+1}{n+1}-\frac{1}{x}\binom{3n+2}{n+1}+\frac{1}{x}\Big[\Big(1-\frac{1}{x}\Big)A_{n+1}(x)+\frac{1}{x}\binom{3n+3}{n+1}\Big].
\end{aligned}$$

Simplifying,

$$\frac{(x-1)^2}{x^2}A_{n+1}(x)=xA_n(x)+\binom{3n+1}{n+1}-\frac{1}{x}\binom{3n+2}{n+1}-\frac{1}{x}\binom{3n+3}{n+1}+\frac{1}{x^2}\binom{3n+3}{n+1}.$$

Therefore

$$A_{n+1}(x)=\frac{x^3}{(x-1)^2}A_n(x)+\frac{(4n+2)x^2-(15n+10)x+9n+6}{2(n+1)(x-1)^2}\binom{3n+1}{n}.$$

Finally,

$$\begin{aligned}
C_{n+1}(x) &= \sum_{j=0}^{2n+2} x^{2n+2-j}\binom{n+1+j}{j} \\
&= \sum_{j=0}^{2n+2} x^{2n+2-j}\binom{n+j}{j}+\sum_{j=1}^{2n+2} x^{2n+2-j}\binom{n+j}{j-1} \\
&= x^2C_n(x)+x\binom{3n+1}{2n+1}+\binom{3n+2}{2n+2} \\
&\quad +\sum_{j=0}^{2n+2} x^{2n+1-j}\binom{n+1+j}{j}-\frac{1}{x}\binom{3n+3}{2n+2} \\
&= x^2C_n(x)+x\binom{3n+1}{2n+1}+\binom{3n+2}{2n+2}+\frac{1}{x}C_{n+1}(x)-\frac{1}{x}\binom{3n+3}{2n+2}
\end{aligned}$$

or

$$C_{n+1}(x)=\frac{x^3}{x-1}C_n(x)+\frac{(2n+2)x^2+(3n+2)x-9n-6}{2(n+1)(x-1)}\binom{3n+1}{n}.$$

3. Applications

Gould [4] collected numerous interesting binomial identities for trigonometric sums and polynomials. We use the formulas given in (3) to obtain identities for sine and cosine sums which we could not locate in Gould's compilation or any other publication.

We set $x = e^{i\theta}$. Then,

$$A_n(x+1) = \sum_{j=0}^{n} \binom{3n-j}{2n} (e^{i\theta}+1)^j = \sum_{j=0}^{n} \binom{3n-j}{2n} \sum_{\nu=0}^{j} \binom{j}{\nu} \big[\cos(\nu\theta) + i\sin(\nu\theta)\big]$$

and

$$B_n(x) = \sum_{j=0}^{n} \binom{3n+1}{n-j} e^{i\theta j} = \sum_{j=0}^{n} \binom{3n+1}{n-j} \big[\cos(j\theta) + i\sin(j\theta)\big].$$

Since $A_n(x+1) = B_n(x)$, we obtain

$$\sum_{j=0}^{n} \binom{3n-j}{2n} \sum_{\nu=0}^{j} \binom{j}{\nu} \cos(\nu\theta) = \sum_{j=0}^{n} \binom{3n+1}{n-j} \cos(j\theta) \tag{4}$$

and

$$\sum_{j=1}^{n} \binom{3n-j}{2n} \sum_{\nu=1}^{j} \binom{j}{\nu} \sin(\nu\theta) = \sum_{j=1}^{n} \binom{3n+1}{n-j} \sin(j\theta). \tag{5}$$

If we apply $C_n(x+1) = D_n(x)$, then we find the following companions of (4) and (5):

$$\sum_{j=0}^{2n} \binom{3n-j}{n} \sum_{\nu=0}^{j} \binom{j}{\nu} \cos(\nu\theta) = \sum_{j=0}^{2n} \binom{3n+1}{n+1+j} \cos(j\theta)$$

and

$$\sum_{j=1}^{2n} \binom{3n-j}{n} \sum_{\nu=1}^{j} \binom{j}{\nu} \sin(\nu\theta) = \sum_{j=1}^{2n} \binom{3n+1}{n+1+j} \sin(j\theta).$$

A theorem of Vietoris (see [7] and [10]) states that if $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$a_0 \ge a_1 \ge \cdots \ge a_n > 0 \quad \text{and} \quad a_{2j} \le \frac{2j-1}{2j} a_{2j-1} \quad (1 \le j \le n/2), \tag{6}$$

then

$$\sum_{j=0}^{n} a_j \cos(j\theta) \quad \text{and} \quad \sum_{j=1}^{n} a_j \sin(j\theta) > 0 \quad (0 < \theta < \pi).$$

A short calculation reveals, that if we set $a_j = \binom{3n+1}{n-j}$ $(j = 0, 1, \ldots, n)$, then (6) is valid. If follows that the sums given in (4) and (5) are positive for all $n \in \mathbb{N}$ and $\theta \in (0, \pi)$.

If we replace in (5) θ by $\pi-\theta$, and add up the two sums on both sides, then we arrive at

$$(7)\quad \sum_{j=1}^{n}\binom{3n-j}{2n}\sum_{\substack{\nu=1\\ \nu\ odd}}^{j}\binom{j}{\nu}\sin(\nu\theta)=\sum_{\substack{j=1\\ j\ odd}}^{n}\binom{3n+1}{n-j}\sin(j\theta)>0 \quad (n\in\mathbb{N};\ 0<\theta<\pi).$$

We obtain similar results if we apply $A_n(x+1)=B_n(x)$ with $x=e^{i\theta}-1$. In particular, we get the following counterpart of (7):

(8)

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{3n+1}{n-j}\sum_{\substack{\nu=1\\ \nu\ odd}}^{j}\binom{j}{\nu}\sin(\nu\theta)=\sum_{\substack{j=1\\ j\ odd}}^{n}\binom{3n-j}{2n}\sin(j\theta)>0\ (n\in\mathbb{N};\ 0<\theta<\pi).$$

Another relative of (7) is given by

(9)

$$\sum_{j=1}^{2n}(-1)^{j-1}\binom{3n+1}{n+1+j}\sum_{\substack{\nu=1\\ \nu\ odd}}^{j}\binom{j}{\nu}\sin(\nu\theta)=\sum_{\substack{j=1\\ j\ odd}}^{2n}\binom{3n-j}{n}\sin(j\theta)>0\ (n\in\mathbb{N};\ 0<\theta<\pi).$$

We set $x=e^{i\theta}-1$ in $C_n(x+1)=D_n(x)$. Then (as before), we replace θ by $\pi-\theta$ and add up. This yields that the two sine polynomials in (9) are equal. To prove the positivity we make use of the known identity

$$(10)\qquad \sum_{\nu=1}^{n}b_\nu\sin((2\nu-1)\theta)=\sum_{k=1}^{n}(b_k-b_{k+1})\frac{\sin^2(k\theta)}{\sin(\theta)}\quad (b_{n+1}=0).$$

Let $b_k=\binom{3n+1-k}{n}$ $(k=1,2,\dots,n)$ and $b_{n+1}=0$. Then, $b_k>b_{k+1}$ $(k=1,\dots,n)$, so that (10) and

$$\sum_{\substack{j=1\\ j\ odd}}^{2n}\binom{3n-j}{n}\sin(j\theta)=\sum_{\nu=1}^{n}b_\nu\sin((2\nu-1)\theta)$$

reveal that the sums in (9) are positive for $n\in\mathbb{N}$ and $\theta\in(0,\pi)$.

We note that the constant lower bound 0, given in (7), (8) and (9), respectively, is best possible.

Additional inequalities for trigonometric polynomials involving binomial coefficients are given in the research papers [1], [2], [3]. Nonnegative trigonometric polynomials have remarkable applications in various branches. For instance, they play an important role in geometric function theory, approximation theory and in the theory of absolutely monotonic functions. More information on this subject can be found in [9, chapter 4].

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