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MINIMAL EXPANSIONS IN REDUNDANT NUMBER SYSTEMS: FIBONACCI BASES AND GREEDY ALGORITHMS

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Dedicated to Helmut Prodinger on the occasion of his 50th birthday

Abstract. We study digit expansions with arbitrary integer digits in base q (q integer) and the Fibonacci base such that the sum of the absolute values of the digits is minimal. For the Fibonacci case, we describe a unique minimal expansion and give a greedy algorithm to compute it. Additionally, transducers to calculate minimal expansions from other expansions are given. For the case of even integer bases q, similar results are given which complement those given in [6].

1. Introduction

We study redundant digit expansions

$$n = \sum_{j \geq 0} \varepsilon_j G_j,$$

where n is an integer, $\varepsilon_j \in \mathbb{Z}$ are arbitrary digits and G_j is the base sequence of the number system. We are interested in minimal expansions with respect to the cost function

$$\sum_{j \geq 0} |\varepsilon_j| \,. \tag{1}$$

This (and a related cost function) has applications in the optimal design of arithmetical hardware (cf. Reitwiesner [11] and Booth [1]), in coding theory (cf. for instance [9]) and in cryptography (cf. Morain and Olivos [10]).

The binary case $G_j = 2^j$ has first been studied by Reitwiesner [11]. The general q-ary case $G_j = q^j$, where $q \geq 2$ is an integer, has been considered in Heuberger and Prodinger [6], where further references for the q-ary case can be found.

It turns out that there is not a unique minimal expansion of a given integer in base q. However, a special expansion has been singled out in that paper [6], which we call the "symmetric signed digit expansion" of n in base q: If q is odd, it is the unique q-ary expansion of n with digits $-(q-1)/2, \ldots, (q-1)/2$. If q is even, it is the unique q-ary expansion with digits $-q/2, \ldots, q/2$ and the additional requirement that if $|\varepsilon_j| = q/2$ for some $j \geq 0$, then $0 \leq \operatorname{sign}(\varepsilon_j)\varepsilon_{j+1} < q/2$. This symmetric signed digit expansion is always a minimal expansion with respect to the costs in (1). Since the symmetric signed digit expansion coincides with the notion of (q, d)-expansions[1] with $d = -(q-1)/2$ if q is odd, we are mostly interested in the case of an even q.

The symmetric signed digit expansion can be calculated by an easy algorithm, which could be described by a deterministic finite transducer, from right to left.[2] Additionally, a formula for

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[1]The (q, d) system is the number system with base q and digits d, $d+1$, …, $d+q-1$, cf. [2].

[2]In this paper, we always assume that digit expansions are written in the form $\ldots\varepsilon_2\varepsilon_1\varepsilon_0$, i.e., with their least significant digit ε_0 right. A transducer which transforms a digit expansion from right to left is therefore a transducer which reads its input as ε_0, ε_1, …, whereas a transducer which transforms a digit expansion from left to right is a transducer which reads its input as ε_ℓ, …, ε_0, where ℓ is the maximal index j such that $\varepsilon_j \neq 0$.

calculating one digit without calculating the others can be given. In Heuberger and Prodinger [7], carry propagations in von Neumann's addition algorithm using the symmetric signed digit expansion have been studied. In Grabner, Heuberger, and Prodinger [3], the frequencies of subblock occurrences in the symmetric signed digit expansion are calculated asymptotically.

It seems natural to ask whether similar properties can be generalized to other number systems, for instance to number systems with non-integer bases or to number systems defined by linear recurrences. As an example for non-integer bases, canonical number systems in the Gaussian integers have been studied in Heuberger [5]. In this case, it is impossible to predict the least significant digit of a minimal expansion from the knowledge of a finite number of digits of the standard expansion.

The simplest example of a number system defined by a recurring sequence is that defined by the Fibonacci numbers. The Zeckendorf expansion [12] can be seen as the standard expansion in this number system. Minimal redundant expansions in this system are the subject of the first part of this paper. We define an "admissible expansion", which is unique (Section 2) and minimal (Section 3) with respect to (1). It can be calculated by a greedy algorithm (Algorithm 1), but it cannot be calculated by a deterministic transducer from right to left. However, it is possible to calculate some minimal expansion from right to left (Section 4). Since number systems defined by recurring sequences are usually related to a greedy algorithm, we also investigate whether we can calculate the admissible expansion from left to right by a transducer. It turns out (Section 5) that the situation is similar to the right-to-left case: There is no transducer to calculate the admissible expansion, but some minimal expansion can be calculated.

At this point, it is a natural question whether these "greedy" and "left-to-right" results can also be ported back to the q-ary case (q even). The affirmative answer (Greedy algorithm in Section 6, Left-right-conversion in Section 8) is given in the second part of this paper. On the way, we also prove a minimality criterion (Theorem 12 in Section 7) for expansions in base q (q even), which also reproves most properties of the symmetric signed digit expansion.

In order to facilitate notation, we declare all definitions and notations to be local with respect to the corresponding part of the paper, i.e., to avoid useless repetitions, we will not always mention "Fibonacci" in the first part of the paper and vice versa.

Part 1. Fibonacci Number System

2. Admissible Expansions

We consider expansions[3] $\boldsymbol{\varepsilon} = (\ldots, \varepsilon_2, \varepsilon_1)$ of integers n in the number system given by the Fibonacci numbers, i. e.

$$n = \sum_{j\geq 1} \varepsilon_j F_j, \qquad \varepsilon_j \in \mathbb{Z},$$

where $\varepsilon_j \neq 0$ for a finite number of $j \geq 1$ and F_j are the Fibonacci numbers[4]

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \in \mathbb{Z} \text{ and } F_0 = 0, F_1 = 1.$$

We are interested in minimal expansions with respect to the costs

$$c(\boldsymbol{\varepsilon}) = \sum_{j\geq 1} |\varepsilon_j|.$$

We remark that

$$F_6 - F_2 = 7 = F_5 + F_3.$$

Since 7 is not a Fibonacci number, both expansions of 7 are minimal with respect to the costs c. Therefore, there is no unique minimal expansion of integers in the Fibonacci system with arbitrary digits.

[3] Boldface letters will be used to denote finite or (formally) infinite sequences. The sequence which consists of zeros only will be written as $\mathbf{0}$.

[4] In the context of digit expansions, the Fibonacci numbers are often defined to start with $F_0 = 1$, $F_1 = 2$. However, for our purposes, it will be convenient to have two base elements 1, this will enable us to consider fewer special cases at the boundary of the expansion.

However, we will describe a special form of minimal expansions.

We say that the finite sequence $(a_r, \ldots, a_1)$ is a subsequence of $\boldsymbol{\varepsilon} \in \mathbb{Z}^{\mathbb{N}}$ if $(a_r, \ldots, a_1) = (\varepsilon_{\ell+r-1}, \ldots, \varepsilon_\ell)$ for some $\ell \geq 1$.

Definition 1. A sequence $\boldsymbol{\varepsilon} = (\ldots, \varepsilon_3, \varepsilon_2, \varepsilon_1) \in \{0, \pm 1\}^{\mathbb{N}}$ with finitely many nonzero elements is called admissible, if

(A1) $\varepsilon_1 = 0$,

(A2) The following sequences (or their negatives) do not occur as subsequences of $\boldsymbol{\varepsilon}$:
  - (a) $(-1, 1)$,
  - (b) $(1, 1)$,
  - (c) $(-1, 0, 1)$,
  - (d) $(1, 0, 1)$,
  - (e) $(1, 0, 0, 1)$.

A sequence $\boldsymbol{\varepsilon}$ is called an admissible expansion of an integer n, if it is an admissible sequence and if $n = \sum_{j \geq 1} \varepsilon_j F_j$.

Theorem 2. *Let n be an integer. Then there is a unique admissible expansion $\boldsymbol{\varepsilon}$ of n. It can be calculated by Algorithm 1.*

This expansion $\boldsymbol{\varepsilon}(n)$ is minimal with respect to the costs c, i. e.,

$$c(\boldsymbol{\varepsilon}(n)) = \min\{c(\boldsymbol{\eta}) : \boldsymbol{\eta} \in \mathbb{Z}^{\mathbb{N}} \text{ is an expansion of } n\}.$$

Algorithm 1 Calculation of the admissible expansion of n.

Input: $n \in \mathbb{Z}$
Output: Admissible expansion $\boldsymbol{\varepsilon} \in \{0, \pm 1\}^{\mathbb{N}}$ of n.
  $\boldsymbol{\varepsilon} := \mathbf{0}$
  $m := n$
  while $m \neq 0$ **do**
    Choose $\ell \geq 2$ such that
$$\left\lfloor \frac{F_{\ell+2} + F_\ell}{5} \right\rfloor < |m| \leq \left\lfloor \frac{F_{\ell+3} + F_{\ell+1}}{5} \right\rfloor.$$
    $\varepsilon_\ell := \operatorname{sign}(m)$
    $m := m - \varepsilon_\ell F_\ell$
  end while

The remainder of this section proves the first part of Theorem 2, i. e., existence, uniqueness and correctness of Algorithm 1.

Lemma 3. *Let $\mathbf{0} \neq \boldsymbol{\eta}$ be an admissible expansion of an integer n and $\ell := \max\{j \geq 1 : \eta_j \neq 0\}$. Then*

$$\left\lfloor \frac{F_{\ell+2} + F_\ell}{5} \right\rfloor < \eta_\ell n \leq \left\lfloor \frac{F_{\ell+3} + F_{\ell+1}}{5} \right\rfloor. \tag{2}$$

Proof. Without loss of generality, we may assume $\eta_\ell = 1$, because $-\boldsymbol{\eta}$ is an admissible expansion of $-n$.

We first consider the case $\eta_j = 0$ for $j \neq \ell$. Since $\ell \geq 2$ by definition, we have $(F_{\ell+2} + F_\ell)/5 < F_\ell = n \leq (F_{\ell+3} + F_{\ell+1})/5$ and the assertion follows immediately. Therefore, we assume in the sequel that there are at least two nonzero digits in the expansion $\boldsymbol{\eta}$. This yields $\ell \geq 5$ by (A2).

Let

$$\alpha := \frac{1 + \sqrt{5}}{2}, \qquad \beta := \frac{1 - \sqrt{5}}{2}.$$

Then we have

$$F_j = \frac{1}{\sqrt{5}} \left(\alpha^j - \beta^j \right) \tag{3}$$

for $j \in \mathbb{Z}$.

We first prove that the sign of any admissible expansion equals the sign of its most significant digit. Indeed, writing $\ell = 2q + r$ for some $r \in \{1, 2\}$ and using (A2a) and (A2b) only, we get

$$\sum_{j=1}^{\ell} \eta_j F_j \geq F_\ell - \sum_{j=0}^{q-1} F_{2j+r} = F_\ell - \sum_{j=0}^{q-1} (F_{2j+r+1} - F_{2j+r-1}) = F_\ell - F_{\ell-1} + F_{r-1} \geq 1.$$

We now write $\ell = 4q + r$ for $2 \leq r \leq 5$. Then using all conditions of (A2) and the above observation, we get

$$n = \sum_{j=1}^{\ell} \eta_j F_j \leq F_\ell + F_{\ell-4} + F_{\ell-8} + \cdots + F_r.$$

Using (3), this can be summed up to obtain

$$n \leq \frac{1}{\sqrt{5}} \left(\alpha^r \frac{\alpha^{4q+4} - 1}{\alpha^4 - 1} - \beta^r \frac{\beta^{4q+4} - 1}{\beta^4 - 1} \right).$$

Since $1/(z^4 - 1) = \frac{1}{5}(z^{-1} + z^{-3})$ for $z = \alpha$, β, we get

$$n \leq \frac{F_{\ell+3} + F_{\ell+1} - F_{r-1} - F_{r-3}}{5}.$$

By construction, the right hand side is an integer. We note that $F_{r-1} + F_{r-3} \in \{1, 2, 3, 4\}$ for $2 \leq r \leq 5$. This yields the upper bound in (2).

Similarly, we can derive a lower bound for n:

$$n \geq F_\ell - (F_{\ell-3} + F_{\ell-7} + \cdots + F_{r'})$$

for $r' = 2 + (\ell - 1) \bmod 4$. Using the above estimate yields

$$n \geq F_\ell - \left\lfloor \frac{F_\ell + F_{\ell-2}}{5} \right\rfloor = \left\lceil \frac{4F_\ell - F_{\ell-2}}{5} \right\rceil = \left\lceil \frac{F_{\ell+2} + F_\ell}{5} \right\rceil.$$

Since the argument of the ceiling function is not an integer (this follows from the derivation of the upper bound above), we get the lower bound in (2). □

We prove the uniqueness of admissible expansions by induction on $|n|$. Since $\lfloor (F_4 + F_2)/5 \rfloor = 0$, there is no nonzero admissible expansion of $n = 0$. For given nonzero n, there is a unique $\ell \geq 2$ and a unique $\eta_\ell \in \{\pm 1\}$ such that (2) holds. Therefore, the digits η_j for $j \geq \ell$ in any admissible expansion of n are uniquely determined. Furthermore, $(\ldots, 0, \eta_{\ell-1}, \ldots, \eta_1)$ is an admissible expansion of $n - \eta_\ell F_\ell$. Since (2) implies

$$-|n| < -\left\lfloor \frac{F_\ell + F_{\ell-2}}{5} \right\rfloor \leq |n| - F_\ell \leq \left\lfloor \frac{F_{\ell-1} + F_{\ell-3}}{5} \right\rfloor < |n|, \tag{4}$$

the remaining digits η_j, $1 \leq j < \ell$, are uniquely determined by induction.

It is clear that the integers $-3 \leq n \leq 3$ have an admissible expansion. Given n with $n \geq 4$, there is an $\ell \geq 5$ such that (2) is satisfied with $\eta_\ell = 1$. The integer $n' := n - F_\ell$ has an admissible expansion $(\ldots, 0, \eta'_{\ell'}, \ldots, \eta'_2, 0)$ by induction. If $n' > 0$, we have $\ell' \leq \ell - 4$ by (4), whereas $\ell' \leq \ell - 3$ if $n' < 0$. In both cases, we may set $\eta_j := \eta'_j$ for $j \neq \ell$ to obtain an admissible expansion of n. This corresponds exactly to Algorithm 1.

This concludes the proof of the first part of Theorem 2.

3. Minimal expansions

We first prove that there are always minimal expansions which only use the digits $0, \pm 1$.

Lemma 4. *For $n \in \mathbb{Z}$, the set $\operatorname{opt}(n)$ of expansions $\boldsymbol{\varepsilon} \in \{0, \pm 1\}^{\mathbb{N}}$ of n with digits $0, \pm 1$ such that*

$$c(\boldsymbol{\varepsilon}) = \min\Big\{ c(\boldsymbol{\eta}) : \boldsymbol{\eta} \in \mathbb{Z}^{\mathbb{N}} \text{ with } n = \sum_{j \geq 1} \eta_j F_j \Big\}.$$

is nonempty.

Proof. It is well known that every positive integer n has a finite expansion $n = \sum_{j=1}^{\ell} \eta_j F_j$ with $\eta_j \in \{0,1\}$. Let now $\boldsymbol{\eta} \in \mathbb{Z}^{\mathbb{N}}$ be a minimal expansion of n with arbitrary integer digits. Among these minimal expansions, choose one such that $\ell := \max\{j : |\eta_j| > 1\}$ is minimal. Among those, choose one such that $|\eta_\ell|$ is minimal. Without loss of generality, we may assume $\eta_\ell > 1$.

It is clear that $\eta_{\ell+1} \leq 0$: Otherwise, we could replace $(\eta_{\ell+2}, \eta_{\ell+1}, \eta_\ell)$ by $(\eta_{\ell+2}+1, \eta_{\ell+1}-1, \eta_\ell-1)$ to get an expansion $\boldsymbol{\eta'}$ of n with $c(\boldsymbol{\eta'}) < c(\boldsymbol{\eta})$. Similarly, we have $\ell \geq 3$ because $2F_2 = 2F_1 = F_3$. We replace $(\eta_{\ell+1}, \eta_\ell, \eta_{\ell-1}, \eta_{\ell-2})$ by $(\eta_{\ell+1}+1, \eta_\ell - 2, \eta_{\ell-1}, \eta_{\ell-2}+1)$ to get an expansion $\boldsymbol{\eta'}$ of n with $c(\boldsymbol{\eta'}) \leq c(\boldsymbol{\eta})$. We have $\max\{j : |\eta'_j| > 1\} \leq \ell$ and $0 \leq \eta'_\ell < \eta_\ell$. This contradicts our choice of $\boldsymbol{\eta}$. □

Lemma 5. *Let n be an integer. Then there is always a minimal expansion $\boldsymbol{\varepsilon} \in \operatorname{opt}(n)$ which satisfies (A2a), (A2b), (A2c), and (A2d).*

Proof. We collect some subsequences which do not occur in any $\boldsymbol{\eta} \in \operatorname{opt}(n)$. To prove this, we also give the replacement which yields lower costs[5].

(5)

Forbidden subsequence	Replacement
$(x,1,1)$	$(x+1,0,0)$
$(-1,1,x)$	$(0,0,x-1)$
$(-1,0,1)$	$(0,-1,0)$
$(0,1,0,1,0,1,0)$	$(1,0,0,0,0,0,-1)$

We say that a sequence $\boldsymbol{\varepsilon} \in \mathbb{Z}^{\mathbb{N}}$ starts with a finite sequence $(a_r, \dots, a_1)$, if $\varepsilon_j = a_j$ for $1 \leq j \leq r$. Note that we always read sequences of digits from right to left.

Any optimal expansion cannot start with the following subsequences:

(6)

Forbidden start	Replacement
$(-1,+1)$	$(0,0)$
$(x,1,0,1)$	$(x+1,0,0,0)$
$(-1,0,0,1)$	$(0,-1,0,0)$

Of course, the above tables also hold when multiplied by -1.

Therefore, all $\boldsymbol{\eta} \in \operatorname{opt}(n)$ satisfy (A2a), (A2b), and (A2c). Assume that $\boldsymbol{\eta} \in \operatorname{opt}(n)$ contains a subsequence $\pm(1,0,1)$. We chose ℓ maximal such that $(\eta_{\ell+2}, \eta_{\ell+1}, \eta_\ell) = \pm(1,0,1)$. Without loss of generality, we consider the case $+(1,0,1)$.

We have $\ell \geq 2$ and $\eta_{\ell-1} = 0$ by (6) and (5). Choose $k \geq 0$ maximal such that

$$(\eta_{\ell+3k+2}, \dots, \eta_{\ell-1}) = (1,0,0)^{(k)} \,\&\, (1,0,1,0),$$

using the following notations: The concatenation $(a_r, \dots, a_1) \,\&\, (b_s, \dots, b_1)$ of two finite sequences is defined to be $(a_r, \dots, a_1, b_s, \dots, b_1)$. For a finite sequence $\boldsymbol{a} = (a_r, \dots, a_1)$, $\boldsymbol{a}^{(k)}$ denotes $\boldsymbol{a} \,\&\, \boldsymbol{a} \,\&\, \cdots \,\&\, \boldsymbol{a}$, where $\boldsymbol{a}$ is repeated k times.

It is clear that $\eta_{\ell+3k+3} = 0$. We have

$$\begin{aligned}(0) \,\&\, (1,0,0)^{(k)} \,\&\, (1,0,1,0) &= (0,1,0)^{(k+1)} \,\&\, (1,0) \leftrightarrow (1,0,-1)^{(k+1)} \,\&\, (1,0)\\ &= (1,0) \,\&\, (-1,1,0)^{(k+1)} \leftrightarrow (1,0) \,\&\, (0,0,-1)^{(k+1)},\end{aligned}$$

where $\leftrightarrow$ means "can be replaced by". This resulting expansion of n is called $\boldsymbol{\eta'}$. We note that $c(\boldsymbol{\eta'}) = c(\boldsymbol{\eta})$. Consider $(\eta'_{\ell+3k+5}, \eta'_{\ell+3k+4}, \eta'_{\ell+3k+3}) = (\eta_{\ell+3k+5}, \eta_{\ell+3k+4}, 1)$. Since $\boldsymbol{\eta'} \in \operatorname{opt}(n)$, we have $\eta'_{\ell+3k+4} = 0$ and $\eta'_{\ell+3k+5} \in \{0,1\}$. By the maximality of k, we get $\eta'_{\ell+3k+5} = 0$. This implies that if $(\eta'_{\ell'+2}, \eta'_{\ell'+1}, \eta'_{\ell'}) = \pm(1,0,1)$ for some ℓ', we have $\ell' \leq \ell - 3$.

The assertion of the lemma follows by induction. □

For brevity, we write $\mathsf{A} := (0,0,1)$, $\bar{\mathsf{A}} := (0,0,-1)$ and write a sequence satisfying (A2a), (A2b), (A2c), and (A2d) just as a juxtaposition of the letters 0, A, $\bar{\mathsf{A}}$. As always, the most significant block is left.

[5]Here and in the sequel, x, x_0, x_1, … denote elements of the actual alphabet.

Proposition 6. *Let n be an integer and $\boldsymbol{\eta} \in \{0, \pm 1\}^{\mathbb{N}}$ be an expansion of n which satisfies (A2a), (A2b), (A2c), and (A2d). Then it is optimal if and only if it does not contain the subsequences $\bar{\mathsf{A}}(\mathsf{A}0)^\ell\mathsf{AA}$ for any $\ell \geq 0$ (or its negatives) and if does not start with $\bar{\mathsf{A}}(\mathsf{A}0)^\ell\mathsf{A}$ for any $\ell \geq 0$ (or its negatives).*

Proof. Assume that $\boldsymbol{\eta} \in \{0, \mathsf{A}, \bar{\mathsf{A}}\}^{\mathbb{N}}$ is optimal. Since $(-1, 0, 0, 1, 0, 0, 1) \leftrightarrow (0, -1, 0, 0, 0, -1, 0)$ and the latter has smaller costs, the block $\bar{\mathsf{A}}\mathsf{AA}$ does not occur in an optimal expansion. The key transformation of the proof of this proposition is

$$0\mathsf{AA} = (0, 0, 0, 1, 0, 0, 1) \leftrightarrow (0, 0, 1, 0, 0, -1, 0) = \mathsf{A}\bar{\mathsf{A}}0.$$

Therefore, for $\ell \geq 1$, $\bar{\mathsf{A}}(\mathsf{A}0)^\ell\mathsf{AA} = \bar{\mathsf{A}}(\mathsf{A}0)^{\ell-1}\mathsf{A}0\mathsf{AA} \leftrightarrow \bar{\mathsf{A}}(\mathsf{A}0)^{\ell-1}\mathsf{AA}\bar{\mathsf{A}}0$, which cannot occur in an optimal expansion by induction.

Similarly, an optimal expansion does not start with $\bar{\mathsf{A}}\mathsf{A}$ (see (6)). Since $F_1 = F_2$, we can replace a start $\bar{\mathsf{A}}(\mathsf{A}0)^{\ell-1}\mathsf{A}0\mathsf{A}$ by $\bar{\mathsf{A}}(\mathsf{A}0)^{\ell-1}\mathsf{AA}0$ for $\ell \geq 1$, and this sequence does not occur. We proved the necessity of the two conditions.

Take any expansion $\boldsymbol{\eta} \in \{0, \mathsf{A}, \bar{\mathsf{A}}\}^{\mathbb{N}}$ of n which satisfies the above conditions. We will transform this expansion to the admissible expansion of n without changing costs. This implies that an optimal expansion that satisfies (A2a), (A2b), (A2c), and (A2d)—such an expansion exists by Lemma 5—has the same costs as the admissible expansion, which is therefore optimal. And this yields optimality for all expansions which satisfy the conditions of the proposition.

Consider the leftmost occurrence of AA in $\boldsymbol{\eta}$, and choose $k \geq 1$, $\ell \geq 0$, and $m \geq 0$ maximal such that it has the form $x_4x_3(\mathsf{A}0)^\ell(\mathsf{AA})^k(\bar{\mathsf{A}}0)^m x_1x_0$. Since k is maximal and since we are considering the leftmost occurrence of AA and since $x_3 = \bar{\mathsf{A}}$ is forbidden, we have $x_3 = 0$. Since ℓ is maximal, $x_4 = 0$ or $x_4 = \bar{\mathsf{A}}$. Since $\mathsf{A}(\bar{\mathsf{A}}0)^m\bar{\mathsf{A}}\bar{\mathsf{A}}$ is forbidden and since m is maximal, we have $x_1x_0 \notin \{\bar{\mathsf{A}}\bar{\mathsf{A}}, \bar{\mathsf{A}}0\}$.

We transform this block as follows:

$$\begin{aligned}
x_4 0(\mathsf{A}0)^\ell(\mathsf{AA})^k(\bar{\mathsf{A}}0)^m x_1x_0 &= x_4(0\mathsf{A})^\ell 0(\mathsf{AA})^k(\bar{\mathsf{A}}0)^m x_1x_0\\
&\leftrightarrow x_4(0\mathsf{A})^\ell(\mathsf{A}\bar{\mathsf{A}})^k 0(\bar{\mathsf{A}}0)^m x_1x_0\\
&= x_4(0\mathsf{A})^{\ell-1}0\mathsf{AA}(\bar{\mathsf{A}}\mathsf{A})^{k-1}(\bar{\mathsf{A}}0)^{m+1}x_1x_0\\
&\leftrightarrow x_4(0\mathsf{A})^{\ell-1}\mathsf{A}\bar{\mathsf{A}}0(\bar{\mathsf{A}}\mathsf{A})^{k-1}(\bar{\mathsf{A}}0)^{m+1}x_1x_0\\
&\leftrightarrow x_4\mathsf{A}(\bar{\mathsf{A}}0)^\ell(\bar{\mathsf{A}}\mathsf{A})^{k-1}(\bar{\mathsf{A}}0)^{m+1}x_1x_0.
\end{aligned}$$

Note that this new block does not contain any occurrence of AA or $\bar{\mathsf{A}}\bar{\mathsf{A}}$ except possibly $x_1x_0 = \mathsf{AA}$. Therefore and since $x_1x_0 \notin \{\bar{\mathsf{A}}0, \bar{\mathsf{A}}\bar{\mathsf{A}}\}$, the transformation did not introduce any forbidden block. Moreover, it did not change the costs. We note that the digits right of $(\mathsf{AA})^k$ have not been changed, they (or some of them) could even be missing (i. e., the block is near the (right) start of the expansion).

We repeat this construction until there is no more occurrence of AA or $\bar{\mathsf{A}}\bar{\mathsf{A}}$ in $\boldsymbol{\eta}$. The resulting sequence is admissible, unless it starts with A (resp. $\bar{\mathsf{A}}$). We choose $\ell \geq 0$ maximal such that the sequence starts with $x_1x_0(\mathsf{A}0)^\ell\mathsf{A}$. By assumption, we have $x_0 \neq \bar{\mathsf{A}}$. Since there is no more block AA in the sequence, we get $x_0 \neq \mathsf{A}$. This implies $x_0 = 0$. Furthermore, $x_1 \neq \mathsf{A}$ by the maximality of ℓ. Using $F_1 = F_2$, we rewrite the start of $\boldsymbol{\eta}$ as $x_1 0(\mathsf{A}0)^\ell\mathsf{A} = x_1(0\mathsf{A})^\ell 0\mathsf{A} = x_1(0\mathsf{A})^\ell\mathsf{A}0$. If $\ell \geq 1$, we introduced a block AA, but we did not introduce a forbidden sequence. Therefore, the above construction can be used once again to remove AA. As we noted above, this does not change the digit η_1.

Finally, this new sequence is admissible. By the argument described above, the proposition is proved. ∎

This concludes the proof of Theorem 2.

4. Calculation of a Minimal Expansion from another Expansion From Right to Left

As in the previous section, $\mathsf{A} = (0, 0, 1)$ and $\bar{\mathsf{A}} = (0, 0, -1)$. We will write $\bar{1} := -1$.

Theorem 7. *Let n be an integer and $\boldsymbol{\eta} = (\eta_\ell, \dots, \eta_1) \in \{0, \pm 1\}^\ell$ be an expansion of n such that $\eta_j \eta_{j+1} = 0$ for $1 \le j \le \ell$ and $\eta_1 = 0$. Then the transducer in Table 1 translates $\boldsymbol{\eta}$ (read from right to left) into a minimal expansion of n.*

	In: 0		In: 01		In: $0\bar{1}$		In: $\perp$	
from	Out:	To:	Out:	To:	Out:	To:	Out:	To:
ε	ε	0	—		—		—	
$\perp$	—		—		—		—	
0	0	0	ε	010	ε	$0\bar{1}0$	$\perp$	$\perp$
010	ε	$\mathsf{A}0$	ε	$10\bar{\mathsf{A}}$	0	$\bar{\mathsf{A}}0$	$\perp\mathsf{A}0$	$\perp$
$10\bar{\mathsf{A}}$	$\bar{\mathsf{A}}$	010	$00\bar{\mathsf{A}}$	10	ε	$\bar{\mathsf{A}}\mathsf{A}0$	$\perp\mathsf{A}0\bar{\mathsf{A}}$	$\perp$
$\bar{\mathsf{A}}\mathsf{A}0$	$\bar{\mathsf{A}}\mathsf{A}0$	0	$\bar{\mathsf{A}}\bar{\mathsf{A}}$	010	$\mathsf{A}0$	$0\bar{1}\bar{\mathsf{A}}$	$\perp\bar{\mathsf{A}}\mathsf{A}0$	$\perp$
$\mathsf{A}0$	$\mathsf{A}0$	0	0	$01\mathsf{A}$	ε	$\bar{\mathsf{A}}\bar{\mathsf{A}}$	$\perp\mathsf{A}0$	$\perp$
$01\mathsf{A}$	ε	$\mathsf{A}\mathsf{A}$	$0\bar{\mathsf{A}}0$	10	A	$\bar{\mathsf{A}}0$	$\perp\mathsf{A}\bar{\mathsf{A}}0$	$\perp$
10	ε	010	00	10	$\bar{\mathsf{A}}$	0	$\perp\mathsf{A}0$	$\perp$
$\mathsf{A}\mathsf{A}$	ε	$\mathsf{A}\bar{\mathsf{A}}0$	A	$01\mathsf{A}$	$\bar{\mathsf{A}}0$	$\bar{\mathsf{A}}0$	$\perp\mathsf{A}\bar{\mathsf{A}}0$	$\perp$
$0\bar{1}0$	ε	$\bar{\mathsf{A}}0$	0	$\mathsf{A}0$	ε	$\bar{1}0\mathsf{A}$	$\perp\bar{\mathsf{A}}0$	$\perp$
$\bar{1}0\mathsf{A}$	A	$0\bar{1}0$	ε	$\mathsf{A}\bar{\mathsf{A}}0$	$00\mathsf{A}$	$\bar{1}0$	$\perp\bar{\mathsf{A}}0\mathsf{A}$	$\perp$
$\mathsf{A}\bar{\mathsf{A}}0$	$\mathsf{A}\bar{\mathsf{A}}0$	0	$\bar{\mathsf{A}}0$	$01\mathsf{A}$	$\mathsf{A}\mathsf{A}$	$0\bar{1}0$	$\perp\mathsf{A}\bar{\mathsf{A}}0$	$\perp$
$\bar{\mathsf{A}}0$	$\bar{\mathsf{A}}0$	0	ε	$\mathsf{A}\mathsf{A}$	0	$0\bar{1}\bar{\mathsf{A}}$	$\perp\bar{\mathsf{A}}0$	$\perp$
$0\bar{1}\bar{\mathsf{A}}$	ε	$\bar{\mathsf{A}}\bar{\mathsf{A}}$	$\bar{\mathsf{A}}$	$\mathsf{A}0$	$0\mathsf{A}0$	$\bar{1}0$	$\perp\bar{\mathsf{A}}\mathsf{A}0$	$\perp$
$\bar{1}0$	ε	$0\bar{1}0$	A	0	00	$\bar{1}0$	$\perp\bar{\mathsf{A}}0$	$\perp$
$\bar{\mathsf{A}}\bar{\mathsf{A}}$	ε	$\bar{\mathsf{A}}\mathsf{A}0$	$\mathsf{A}0$	$\mathsf{A}0$	$\bar{\mathsf{A}}$	$0\bar{1}\bar{\mathsf{A}}$	$\perp\bar{\mathsf{A}}\mathsf{A}0$	$\perp$

TABLE 1. Transducer to compute a minimal expansion from right to left. Initial state ε, terminal state $\perp$. ε denotes the empty word. $\perp$ denotes the left end of the sequence.

Corollary 8. *Let n be a nonnegative integer and $\boldsymbol{\eta} = (\eta_\ell, \dots, \eta_1) \in \{0, 1\}^\ell$ be its Zeckendorf expansion. Then the transducer in Figure 1 translates $\boldsymbol{\eta}$ (read from right to left) into a minimal expansion of n.*

Proof of Corollary 8. If the edges with input label $0\bar{1}$ are removed from the transducer in Table 1, only the states ε, 0, 010, $10\bar{\mathsf{A}}$, $\mathsf{A}0$, $01\mathsf{A}$, 10, $\mathsf{A}\mathsf{A}$, and $\mathsf{A}\bar{\mathsf{A}}0$ are accessible. When leaving $10\bar{\mathsf{A}}$, the first letter of the output is always $\bar{\mathsf{A}}$, therefore we can output $\bar{\mathsf{A}}$ already on the way to $10\bar{\mathsf{A}}$, which is then equivalent to 10 and can be removed. Similarly, a symbol 0 is output when leaving $\mathsf{A}0$ and the word $\bar{\mathsf{A}}0$ is output when leaving $\mathsf{A}\bar{\mathsf{A}}0$. Therefore, these two states can be replaced by a new state with label A. The remaining transducer corresponds to Figure 1. $\square$

Proof of Theorem 7. For each edge $\boldsymbol{a} \xrightarrow{\boldsymbol{i}|\boldsymbol{o}} \boldsymbol{b}$, we can check that $\boldsymbol{i} \,\&\, \boldsymbol{a} \leftrightarrow \boldsymbol{b} \,\&\, \boldsymbol{o}$, which implies that the resulting expansion is indeed an expansion of n. Since the output consists of 0, A, and $\bar{\mathsf{A}}$ only, conditions (A2a), (A2b), (A2c), and (A2d) are automatically satisfied.

We consider the output of the transducer in Table 1. To describe this language, we construct a new "output automaton" just by replacing the input by the output. We expand edges where necessary: For instance, we replace $10\bar{\mathsf{A}} \xrightarrow{00\bar{\mathsf{A}}} 10$ by $10\bar{\mathsf{A}} \xrightarrow{\bar{\mathsf{A}}} v_1 \xrightarrow{0} v_2 \xrightarrow{0} 10$ for some new states v_1 and v_2. The result is a non deterministic finite automaton. We use the standard algorithm to transform it to a deterministic finite automaton. Although in theory, this may be an exponential process, it turns out that in our case, many states are either not accessible or can be identified with other states. Therefore, we just number the new states without retaining their actual meanings. We just remember that state 1 corresponds to the old initial state. The resulting automaton is described in Table 2.

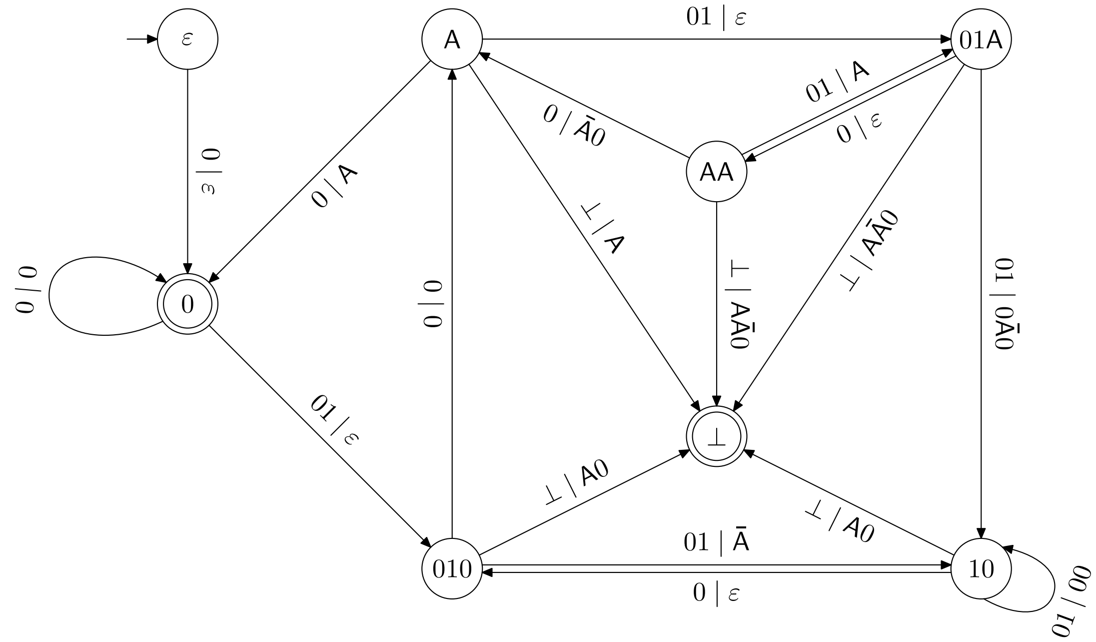

FIGURE 1. Transducer to compute a minimal expansion from the Zeckendorf expansion from right to left. Initial state ε, terminal states 0 and $\perp$. ε denotes the empty word. $\perp$ denotes the left end of the sequence.

We make the following observations:

After reading ...	...the set of possible states is
A	$\{3, 8, 14, 15, 21\}$,
AA	$\{3\}$,
0AA	$\{10\}$.

Since there is no edge $\bar{\mathsf{A}}$ leaving state 3 and no edge A leaving state 10, the output of the transducer in Table 1 does not contain the subsequences $\bar{\mathsf{A}}(\mathsf{A}0)^\ell\mathsf{AA}$ for $\ell \geq 0$.

Similarly, we note that if we start at state 1, the words $\bar{\mathsf{A}}\mathsf{A}$ and $\mathsf{A}0\mathsf{A}$ cannot be read, we see that the output of the transducer in Table 1 does not start with $\bar{\mathsf{A}}(\mathsf{A}0)^\ell\mathsf{A}$ for $\ell \geq 0$. Since the transducer in Table 1 is invariant on multiplication by -1, the same holds for the corresponding negative sequences. By Proposition 6, the output of the Transducer in Table 1 is a minimal expansion of n. □

Remark 9. The output of the transducer in Table 1 is in fact the language defined by the automaton in Figure 2.

Proof. We can identify some states of Table 2:

Identified states	New name
$1, 14, 18$	31
$2, 7, 9, 12, 13, 17, 23, 24, 25, 26$	32
$6, 16, 19$	27
$8, 15, 21$	28
$10, 22$	30
$11, 20$	29.

The result corresponds to Figure 2. □

from	In: 0	A	Ā	⊥
1	2	3	4	5
2	7	8	6	5
3	10	3	—	—
4	11	—	4	—
5	—	—	—	—
6	13	14	4	5
7	7	15	16	5
8	9	3	18	5
9	25	15	16	5
10	20	—	19	—
11	22	21	—	—
12	12	15	16	5
13	23	15	16	5
14	24	3	4	5
15	17	3	18	5
16	12	14	4	5
17	17	15	16	5
18	26	3	4	5
19	23	14	4	5
20	10	15	—	—
21	25	3	18	5
22	11	—	16	—
23	13	15	16	5
24	25	8	16	5
25	9	15	16	5
26	23	15	6	5

TABLE 2. Automaton describing the output of the transducer in Table 1. Initial state 1. Terminal state 5.

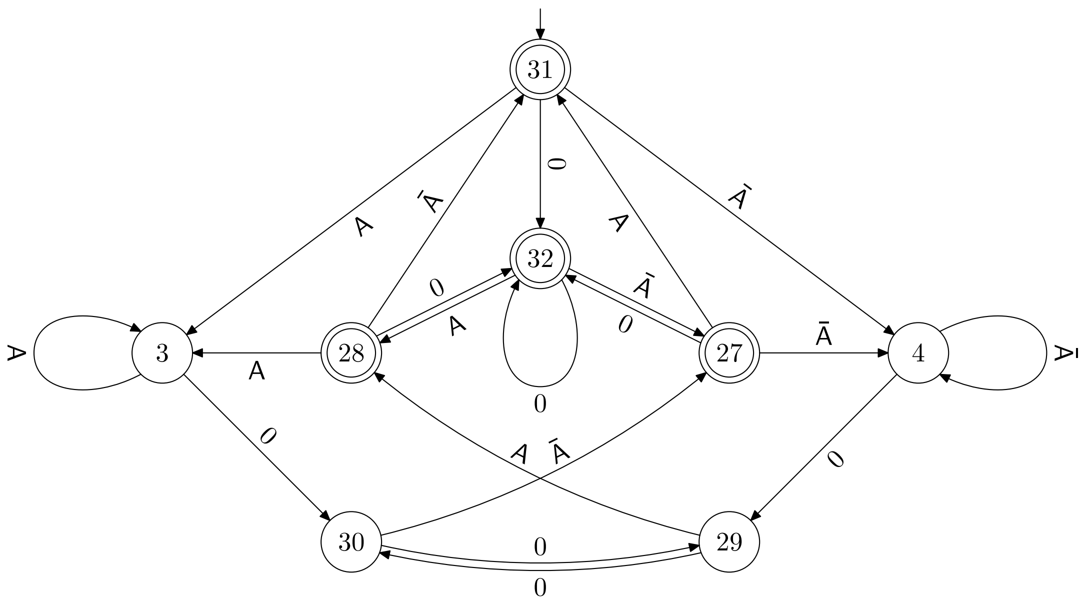

FIGURE 2. Output of the transducer in Table 1. Initial state 31. Terminal states 27, 28, 31, 32.

5. Calculation of a Minimal Expansion from another Expansion From Left to Right

It is also possible to calculate a minimal expansion from left to right, i. e., from the most significant digits to the least significant digits. We write $\bar{1} = -1$, $\mathsf{C} := (1, 0, 0)$, $\bar{\mathsf{C}} := (-1, 0, 0)$. In this case, we have to be more careful about the boundary conditions, especially on the right start of the expansion.

We cannot expect to calculate an admissible expansion by a deterministic finite transducer: $\boldsymbol{\alpha} := (0\mathsf{C})^\ell 0\mathsf{C}0000$ is an admissible expansion. However, the expansion $\boldsymbol{\beta} := (0\mathsf{C})^\ell 0\mathsf{C}\mathsf{C}0$ (whose left $4\ell + 4$ digits are identical to those of $\boldsymbol{\alpha}$) has to be rewritten to $\mathsf{C}(\bar{\mathsf{C}}0)^{\ell+1}0$. Therefore, the most significant digit of an admissible expansion cannot be calculated from the knowledge of the k most significant digits in (for instance) the Zeckendorf expansion for any absolute constant k. However, a minimal expansion can be calculated, as it is demonstrated in Theorem 10.

Theorem 10. *Let n be an integer and $\boldsymbol{\eta} = (\eta_\ell, \ldots, \eta_1) \in \{0, \pm 1\}^\ell$ be an expansion of n such that $\eta_j \eta_{j+1} = 0$ for $j \geq 1$ and $\eta_1 = 0$ and $\eta_\ell = 0$. Then the transducer in Table 3 translates $\boldsymbol{\eta}$ (read from left to right) into a minimal expansion of n.*

	In: 0		In: 10		In: $\bar{1}0$		In: $\perp$	
from	Out:	To:	Out:	To:	Out:	To:	Out:	To:
ε	ε	0	—		—		—	
$\perp$	—		—		—		—	
0	0	0	ε	010	ε	$0\bar{1}0$	$0\perp$	$\perp$
01	ε	010	C	0	00	01	$01\perp$	$\perp$
010	ε	$0\mathsf{C}$	ε	$\mathsf{C}0\bar{1}$	0	$0\mathsf{C}$	$010\perp$	$\perp$
$0\mathsf{C}$	$0\mathsf{C}$	0	ε	$0\mathsf{C}10$	0	$0\mathsf{C}1$	$0\mathsf{C}\perp$	$\perp$
$0\mathsf{C}1$	ε	$0\mathsf{C}10$	C	$0\bar{\mathsf{C}}$	$0\mathsf{C}0$	01	$0\mathsf{C}1\perp$	$\perp$
$0\mathsf{C}10$	ε	$\mathsf{C}\bar{\mathsf{C}}0$	C	$0\bar{\mathsf{C}}\bar{1}$	$0\mathsf{C}$	$0\mathsf{C}$	$0\mathsf{C}10\perp$	$\perp$
$\mathsf{C}\bar{\mathsf{C}}0$	$\mathsf{C}\bar{\mathsf{C}}0$	0	$\mathsf{C}\bar{\mathsf{C}}$	010	$0\mathsf{C}$	$0\mathsf{C}1$	$\mathsf{C}\bar{\mathsf{C}}0\perp$	$\perp$
$\mathsf{C}0\bar{1}$	C	$0\bar{1}0$	$\mathsf{C}00$	$0\bar{1}$	ε	$\mathsf{C}\bar{\mathsf{C}}0$	$\mathsf{C}0\bar{1}\perp$	$\perp$
$0\bar{1}$	ε	$0\bar{1}0$	00	$0\bar{1}$	$\bar{\mathsf{C}}$	0	$0\bar{1}\perp$	$\perp$
$0\bar{1}0$	ε	$0\bar{\mathsf{C}}$	0	$0\bar{\mathsf{C}}$	ε	$\bar{\mathsf{C}}01$	$0\bar{1}0\perp$	$\perp$
$0\bar{\mathsf{C}}$	$0\bar{\mathsf{C}}$	0	0	$0\bar{\mathsf{C}}\bar{1}$	ε	$0\bar{\mathsf{C}}\bar{1}0$	$0\bar{\mathsf{C}}\perp$	$\perp$
$0\bar{\mathsf{C}}\bar{1}$	ε	$0\bar{\mathsf{C}}\bar{1}0$	$0\bar{\mathsf{C}}0$	$0\bar{1}$	$\bar{\mathsf{C}}$	$0\mathsf{C}$	$0\bar{\mathsf{C}}\bar{1}\perp$	$\perp$
$0\bar{\mathsf{C}}\bar{1}0$	ε	$\bar{\mathsf{C}}\mathsf{C}0$	$0\bar{\mathsf{C}}$	$0\bar{\mathsf{C}}$	$\bar{\mathsf{C}}$	$0\mathsf{C}1$	$0\bar{\mathsf{C}}\bar{1}0\perp$	$\perp$
$\bar{\mathsf{C}}\mathsf{C}0$	$\bar{\mathsf{C}}\mathsf{C}0$	0	$0\bar{\mathsf{C}}$	$0\bar{\mathsf{C}}\bar{1}$	$\bar{\mathsf{C}}\mathsf{C}$	$0\bar{1}0$	$\bar{\mathsf{C}}\mathsf{C}0\perp$	$\perp$
$\bar{\mathsf{C}}01$	$\bar{\mathsf{C}}$	010	ε	$\bar{\mathsf{C}}\mathsf{C}0$	$\bar{\mathsf{C}}00$	01	$\bar{\mathsf{C}}01\perp$	$\perp$

Table 3. Transducer to compute a minimal expansion from left to right. Initial state ε, terminal state $\perp$. ε denotes the empty word. $\perp$ denotes the right end of the sequence. It is assumed that the most significant input digit is 0, i. e., we start at least one position left of the "true" (nonzero) most significant digit of the input expansion.

Proof. The proof is analogous to the proof of Theorem 7. The automaton describing the output of the transducer in Table 3 is given in Table 4. It can easily be checked that the sequences $\bar{\mathsf{C}}\mathsf{C}\mathsf{C}$, $\bar{\mathsf{C}}\mathsf{C}0\mathsf{C}\mathsf{C}$, $\bar{\mathsf{C}}\mathsf{C}0\mathsf{C}0\mathsf{C}$, $\bar{\mathsf{C}}\mathsf{C}1$, $\bar{\mathsf{C}}\mathsf{C}0\mathsf{C}1$, $\bar{\mathsf{C}}1$, $\bar{\mathsf{C}}\mathsf{C}01$, and $\bar{\mathsf{C}}\mathsf{C}0\mathsf{C}01$ are forbidden by the automaton in Table 4. Therefore, the output of the transducer in Table 3 is a minimal expansion of n by Proposition 6. ∎

from	In: 0	C	$\bar{\mathsf{C}}$	1	$\bar{1}$	$\perp$
1	3	4	2	—	—	—
2	5	6	—	—	—	—
3	8	9	10	11	11	7
4	13	—	12	—	—	—
5	15	16	—	14	—	—
6	17	—	12	—	—	—
7	—	—	—	—	—	—
8	8	18	19	11	11	7
9	21	4	12	11	—	7
10	23	6	2	—	11	7
11	24	—	—	—	—	—
12	25	6	—	—	—	—
13	27	—	26	—	14	—
14	24	—	—	—	—	7
15	5	18	—	—	—	—
16	28	4	12	14	—	7
17	29	4	19	—	14	7
18	30	4	12	14	—	7
19	31	6	2	—	14	7
20	21	18	19	11	11	7
21	20	18	19	11	14	7
22	23	18	10	11	11	7
23	28	18	19	14	11	7
24	—	—	—	—	—	7
25	22	18	2	14	—	7
26	20	6	2	—	14	7
27	13	—	19	—	—	—
28	23	18	19	11	11	7
29	21	9	19	11	11	7
30	31	18	19	11	14	7
31	30	18	19	14	11	7

TABLE 4. Automaton describing the output of the transducer in Table 3. Initial state 1. Terminal state 7.

Part 2. Symmetric Signed Digit Expansion in Base q

6. Greedy Algorithm

In this part of the paper, we consider the symmetric signed digit expansion defined in [6]: Let $q \geq 2$ be an even integer. There is a unique expansion $n = \sum_{j\geq 0} \varepsilon_j q^j$ with $\boldsymbol{\varepsilon} \in \{-q/2, \dots, q/2\}^{\mathbb{N}_0}$ and finitely many nonzero digits such that $|\varepsilon_j| = q/2$ implies $0 \leq \operatorname{sign}(\varepsilon_j)\varepsilon_{j+1} < q/2$.

Proposition 11. *Let $q \geq 2$ be even, $0 \neq n$ be an integer, and $\boldsymbol{\varepsilon} \in \{-q/2, \dots, q/2\}^{\mathbb{N}_0}$ its symmetric signed digit expansion to base q. Let $\ell := \max\{j : \varepsilon_j \neq 0\}$. Then*

$$\ell = \left\lfloor \log_q \frac{2(q+1)\,|n|}{q+2} \right\rfloor, \tag{7}$$

$$\varepsilon_\ell = \operatorname{sign}(n) \left\lfloor \frac{|n|}{q^\ell} + \frac{q}{2(q+1)} \right\rfloor. \tag{8}$$

Proof. We first define the following quantities for $k \geq 0$:[6]

$$C(k) := \frac{q}{2} \cdot q^k + \left(\frac{q}{2} - 1\right) q^{k-1} + \frac{q}{2} \cdot q^{k-2} + \left(\frac{q}{2} - 1\right) q^{k-3} + \cdots + \left(\frac{q}{2} - [k \text{ odd}]\right) \cdot 1,$$
$$D(k) := \left(\frac{q}{2} - 1\right) q^k + \frac{q}{2} \cdot q^{k-1} + \left(\frac{q}{2} - 1\right) q^{k-2} + \frac{q}{2} \cdot q^{k-3} + \cdots + \left(\frac{q}{2} - [k \text{ even}]\right) \cdot 1.$$

We calculate that

$$C(k) = \frac{q^{k+1}(q+2)}{2(q+1)} - \frac{1}{2} - \frac{(-1)^{k+1}}{2(q+1)}.$$

Since $C(k)$ is an integer by construction and $0 < 1/2 + (-1)^{k+1}/(2q+2) < 1$, we see that

$$C(k) = \left\lfloor \frac{q^{k+1}(q+2)}{2(q+1)} \right\rfloor.$$

Similarly, we get

$$D(k) = \left\lfloor \frac{q^{k+2}}{2(q+1)} \right\rfloor.$$

We easily check that

$$C(k-1) + D(k-1) + 1 = q^k \text{ and } D(k-1) + \frac{q}{2} q^k = C(k). \tag{9}$$

Now, let $\boldsymbol{\varepsilon}$ be the symmetric signed digit expansion of the integer n and ε_ℓ be the most significant digit. Without loss of generality, we may assume $n > 0$.

Let $r := \ell \bmod 2$. Since $(q/2-1)q + q/2 < (q/2)q + (q/2-1)$ and since there are no two consecutive digits with absolute value $q/2$, we have

$$\begin{aligned} n &= \varepsilon_\ell q^\ell + \sum_{j=0}^{(\ell-r)/2-1} (\varepsilon_{2j+r+1} q + \varepsilon_{2j+r}) q^{2j+r} + [\ell \text{ odd}]\, \varepsilon_0 \\ &\leq \varepsilon_\ell q^\ell + \sum_{j=0}^{(\ell-r)/2-1} \left(\frac{q}{2} \cdot q + \left(\frac{q}{2} - 1\right)\right) q^{2j+r} + [\ell \text{ odd}]\, \frac{q}{2} = \varepsilon_\ell q^\ell + C(\ell-1). \end{aligned}$$

Since $C(\ell-1) < q^\ell$ and $n > 0$, we conclude that $\varepsilon_\ell \geq 0$. By definition, ε_ℓ is nonzero, therefore, we have $\varepsilon_\ell \geq 1$. This implies that $\varepsilon_{\ell-1} > -q/2$. We can derive a lower bound $n \geq \varepsilon_\ell q^\ell - D(\ell-1)$. Combining the two bounds and using (9), we obtain

$$\varepsilon_\ell q^\ell - D(\ell-1) \leq n < (\varepsilon_\ell + 1) q^\ell - D(\ell-1). \tag{10}$$

This is equivalent to

$$\varepsilon_\ell q^\ell \leq n + \frac{q^{\ell+1}}{2(q+1)} < (\varepsilon_\ell + 1) q^\ell.$$

It is easily seen that this implies (8).

If $\varepsilon_\ell = q/2$, we have $\varepsilon_{\ell-1} < q/2$, and the upper bound in (10) can be sharpened to

$$n < \varepsilon_\ell q^\ell + D(\ell-1) + 1 = \frac{q}{2} q^\ell + D(\ell-1) + 1 = C(\ell) + 1 = q^{\ell+1} - D(\ell). \tag{11}$$

Since $1 \leq \varepsilon_\ell \leq q/2$, inequalities (10) and (11) can be combined to give

$$\left\lceil \frac{q^\ell(q+2)}{2(q+1)} \right\rceil = q^\ell - D(\ell-1) \leq n < q^{\ell+1} - D(\ell) = \left\lceil \frac{q^{\ell+1}(q+2)}{2(q+1)} \right\rceil.$$

This is equivalent to (7). ☐

Proposition 11 enables us to give a greedy algorithm to compute the symmetric signed digit expansion.

[6] We use Iverson's notation: $[P] = 1$ if condition P is true, 0 otherwise, compare [4].

Algorithm 2 Greedy Algorithm to Compute the Symmetric Signed Digit Expansion

Input: $q \geq 2$ even, n an integer
Output: Symmetric signed digit expansion $\boldsymbol{\varepsilon}$ of n.

$\boldsymbol{\varepsilon} := \mathbf{0}$
$m := n$
while $m \neq 0$ **do**
  $\ell := \left\lfloor \log_q \frac{2(q+1)|m|}{q+2} \right\rfloor$
  $\varepsilon_\ell := \operatorname{sign}(m) \left\lfloor \frac{|m|}{q^\ell} + \frac{q}{2(q+1)} \right\rfloor$
  $m := m - \varepsilon_\ell q^\ell$
end while

7. Minimality of Expansions to Base q

Theorem 12. *Let $q \geq 2$ be even and $\boldsymbol{\varepsilon} \in \mathbb{Z}^{\mathbb{N}_0}$ be an expansion of an integer n. This expansion is a minimal expansion of n, i. e.,*

$$\sum_{j\geq 0} |\varepsilon_j| = \min \left\{ \sum_{j\geq 0} |\eta_j| : \boldsymbol{\eta} \in \mathbb{Z}^{\mathbb{N}_0} \text{ and } \sum_{j\geq 0} \eta_j q^j = n \right\},$$

if and only if $|\varepsilon_j| \leq q/2$ for $j \geq 0$ and if it does not contain the following subsequences (or their negatives):

1. *$(q/2, q/2) \,\&\, (q/2-1, q/2)^{(\ell)} \,\&\, (q/2)$ for $\ell \geq 0$,*
2. *$(x) \,\&\, (q/2-1, q/2)^{(\ell)} \,\&\, (q/2)$ for $x < 0$ and $\ell \geq 0$.*

Proof. We first prove necessity. Let $\boldsymbol{\varepsilon} \in \mathbb{Z}^{\mathbb{N}_0}$ be a minimal expansion of n. If $\varepsilon_j > q/2$, we can replace $(\varepsilon_{j+1}, \varepsilon_j)$ by $(\varepsilon_{j+1}+1, \varepsilon_j - q)$. If $\varepsilon_j \geq q$, then $|\varepsilon_j - q| = \varepsilon_j - q = |\varepsilon_j| - q$. If $q/2+1 \leq \varepsilon_j < q$ then $|\varepsilon_j - q| \leq q/2 - 1 \leq |\varepsilon_j| - 2$. Therefore, the original $\boldsymbol{\varepsilon}$ was not minimal. Therefore, all minimal expansions have digits of absolute value at most $q/2$.

Next, we note that $(x, q/2, q/2, q/2) \leftrightarrow (x+1, -q/2+1, -q/2+1, -q/2)$ for all $x \in \mathbb{Z}$. The latter has smaller cost. Similarly, $(x, q/2) \leftrightarrow (x+1, -q/2)$, which is less expensive for $x < 0$. Finally, $(q/2-1, q/2, q/2) \leftrightarrow (q/2, -q/2+1, -q/2)$ without changing costs. Inductively, this shows that the two subsequences cannot occur in an optimal expansion.

We now turn to the proof of sufficiency. Let $\boldsymbol{\varepsilon} \in \{-q/2, \ldots, q/2\}^{\mathbb{N}_0}$ be an expansion of n that does not contain the two subsequences described in the theorem. We claim that we can transform it to an admissible expansion of n without changing costs. We consider the rightmost occurrence of $q/2, q/2$ in $\boldsymbol{\varepsilon}$ and choose ℓ maximal such that it has the form $(x_2, x_1) \,\&\, (q/2, q/2-1)^{(\ell)} \,\&\, (q/2, q/2, x_0)$ for some x_0, x_1, x_2. We note that $|x_0| < q/2$ since the subsequences $(q/2, q/2, q/2)$ and $(q/2, -q/2)$ are forbidden. Similarly, we have $0 \leq x_1 \leq q/2-1$. If $x_1 = q/2-1$, the maximality of ℓ implies $0 \leq x_2 \leq q/2-1$. We have

$$\begin{aligned}(x_2, x_1) \,\&\, \left(\frac{q}{2}, \frac{q}{2}-1\right)^{(\ell)} \,\&\, \left(\frac{q}{2}, \frac{q}{2}, x_0\right) &\leftrightarrow (x_2, x_1+1) \,\&\, \left(-\frac{q}{2}+1, -\frac{q}{2}\right)^{(\ell)} \,\&\, \left(-\frac{q}{2}+1, -\frac{q}{2}, x_0\right) \\ &= (x_2, x_1+1) \,\&\, \left(-\frac{q}{2}+1, -\frac{q}{2}\right)^{(\ell+1)} \,\&\, (x_0).\end{aligned}$$

We note that this transformation did not change costs (since $x_1 \geq 0$) and all new digits are in the range $-q/2, \ldots, q/2$. Furthermore, no subsequence $\pm(q/2, q/2)$ occurs since $x_1 + 1 = q/2$ implies $0 \leq x_2 \leq q/2-1$. Similarly, no subsequence $\pm(x, q/2)$ with $x < 0$ occurs in the transformed part of the expansion. This and the fact that there is no block $\pm(q/2, q/2)$ right of the considered block imply that the transformation did not introduce a forbidden block. However, one occurrence of $(q/2, q/2)$ has been removed without changing costs.

By induction, we get an admissible expansion of n without changing costs. Since each integer n has a unique admissible expansion (by [6] or by Proposition 11), we see that the admissible expansion is minimal (we reproved this part of Theorem 2 of [6]) and therefore, all expansions which satisfy the conditions of the theorem are minimal. □

8. Calculation of a minimal expansion from left to right in base q

We consider first the case of an even base $q \geq 4$. It is not surprising that it is impossible to calculate the admissible expansion from left to right, since

$$\left(0, \frac{q}{2}, \frac{q}{2}-1, \frac{q}{2}, \frac{q}{2}-1, \ldots, \frac{q}{2}, \frac{q}{2}-1\right)$$

is admissible, whereas

$$\left(0, \frac{q}{2}, \frac{q}{2}-1, \frac{q}{2}, \frac{q}{2}-1, \ldots, \frac{q}{2}, \frac{q}{2}\right)$$

has to be transformed to

$$\left(1, -\frac{q}{2}+1, -\frac{q}{2}, -\frac{q}{2}+1, -\frac{q}{2}, \ldots, -\frac{q}{2}+1, -\frac{q}{2}\right).$$

Therefore, we can only calculate some minimal expansion from an arbitrary expansion in base q with digits $-q/2, \ldots, q/2$ from left to right. This can be done using a transducer. In theory, such a transducer will have many states since its input alphabet $-q/2, \ldots, q/2$ may become rather large. The transducer is described in Table 5 in abbreviated form. We consider sets A, B, E, and F as described in the table. A lowercase letter $a, \ldots, f$ will always designate an element of the corresponding set. Overlined symbols are written for their negatives.

Look at the fifth row "5 $e\bar{c}\bar{d}$" and the sixth column "$e\bar{c}$ $\bar{d}c$ -4". This information is relevant if we are in a state (x, y, z) with an $x \in E$, $y = -q/2+1$ and $z = -q/2$ and we read a further digit $w = c = q/2-1$. The entry says that we have to write immediately $e\bar{c}$, i.e., (x, y), and that our next state is $\bar{d}c$, which corresponds to the negative of row 4. Indeed, the forth state is labelled $e\bar{c}$. Since $d = q/2 \in E$, this corresponds to our new situation.

Of course, we sometimes write different symbols: For instance in row 5 and column 1, we are in a state (x, y, z) with $x \in E$ etc. We write $(x-1, q/2)$ and continue with cd.

As stated in Table 5, the transducer does not have proper initial and terminal states. The following convention is used: We start reading one digit left of the "true" (i. e., nonzero) most significant digit and are in state 0. At the right end of the input sequence, we add a further (fictive) 0, follow the transducer this one more step, and arrive in state 0. Here, we can just drop the superfluous 0 and are done.

Theorem 13. *Let $q \geq 4$ be even, n be an integer and $\boldsymbol{\eta} = (\eta_\ell, \ldots, \eta_0) \in \{-q/2, \ldots, q/2\}^{\ell+1}$ be an expansion of n in base q. Then the transducer in Table 5 translates $\boldsymbol{\eta}$ (read from left to right) into a minimal expansion of n.*

Proof. First we easily check that the transducer indeed produces an expansion of n. By Theorem 12, minimality is equivalent to the absence of some forbidden subsequences. To prove that some subsequences do not occur, we could split up some of the states—this would be necessary in order to be able to distinguish between some "normal" element $e \in E$ and a "dangerous" $q/2$ (which is also contained in E)—and consider the output automaton as in several other places in this paper.

However, it seems to be more manageable to do a direct verification of the 35 transition rules using the following lemma:

Lemma 14. *Let $q \geq 4$ be even and $\boldsymbol{\varepsilon}$, $\boldsymbol{\eta} \in \{-q/2, \ldots, q/2\}^{\mathbb{N}_0}$ be two expansions of the same integer n in base q. Assume that there is an integer k such that $\varepsilon_j = \eta_j$ for all $j > k$. Then*

1. *$|\varepsilon_k - \eta_k| \leq 1$.*
2. *If $\eta_k = \varepsilon_k + 1$ then $\eta_{k-1} \leq -q/2+1 < 0$ and $\varepsilon_{k-1} \geq q/2-1 > 0$.*

Proof. Without loss of generality, we assume $\eta_k > \varepsilon_k$. Then

$$0 = \frac{1}{q^k}\left(\sum_{j\geq 0}\eta_j q^j - \sum_{j\geq 0}\varepsilon_j q^j\right) = \sum_{j=0}^{k}(\eta_j - \varepsilon_j)q^{j-k} = (\eta_k - \varepsilon_k) + \sum_{j=1}^{k}\frac{\eta_{k-j}-\varepsilon_{k-j}}{q^j}$$
$$> \eta_k - \varepsilon_k + \frac{\eta_{k-1}-\varepsilon_{k-1}}{q} + \sum_{j=2}^{\infty}\frac{-q/2-q/2}{q^j} = \eta_k - \varepsilon_k + \frac{\eta_{k-1}-\varepsilon_{k-1}}{q} - \frac{1}{q-1}.$$

This implies

$$\eta_k - \varepsilon_k < 1 + \frac{1}{q-1}$$

and the first assertion of the lemma follows.

If $\eta_k - \varepsilon_k = 1$, we get

$$\frac{\eta_{k-1}-\varepsilon_{k-1}}{q} < -\frac{q-2}{q-1},$$

which yields

$$\eta_{k-1} - \varepsilon_{k-1} \leq -q+1.$$

The second assertion follows. □

As an example, we consider the transition from state $e\bar{c}\bar{d}$ with input d. According to Table 5, we write $e\bar{c}$ and go to state $\bar{c}\bar{d}$. It is clear that this operation cannot produce the first forbidden pattern in Theorem 12. On the other hand, the second pattern could be matched if the pending digits $\bar{c}\bar{d}\ldots$ would be changed to $\bar{d}(\bar{c}\bar{d})^s\bar{d}$ for some $s \geq 0$ by the (unknown) forthcoming digits. However, this is impossible by Lemma 14: decreasing $\bar{c}$ to $\bar{d}$ implies that the next digit is positive. □

We remark that the output of the transducer in Table 5 depends on the specific input sequence: The input $0d\bar{c}\bar{d}\bar{c}d0$ gives the output $0cdd(\overline{c-1})\bar{d}0$, whereas the equivalent input $0d\bar{c}\bar{d}(\overline{c-1})\bar{d}0$ remains unchanged.

Now, we turn to the case $q = 2$.

Theorem 15. *Let $q = 2$, n be an integer and $\boldsymbol{\eta} = (\eta_\ell, \ldots, \eta_0) \in \{-1, 0, 1\}^{\ell+1}$ be an expansion of n. Then the transducer in Table 6 translates $\boldsymbol{\eta}$ (read from left to right) into a minimal expansion of n.*

Proof. The output automaton is given in Table 7. It does not allow subsequences 111, 11011, 110101, $\bar{1}1$, $\bar{1}011$, $\bar{1}0101$ (and their negatives). Hence the output is minimal by Theorem 12. □

We remark that as in the case of $q \geq 4$, the transducer in Table 6 does not define a unique minimal expansion: The input $01101\bar{1}00$ is translated to $10\bar{1}00100$, whereas the (equivalent) input 01100100 is not changed.

Furthermore, if we restrict the input to (nonnegative) binary expansions, it is useful to replace the transition $0110 \xrightarrow{0|0110} 0$ by $0110 \xrightarrow{0|10\bar{1}0} 0$. Then the transducer in Table 6 can be simplified to the transducer in Figure 3. We note that the algorithm of Joye and Yen [8] leads to the same transducer.

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from		In: $\bar d$			In: $\bar c$			In: $\bar a$			In: 0			In: a			In: c			In: d		
state	pattern	Out	queue	To																		
1	0	ε	$0\bar d$	-3	0	$\bar c$	-2	0	$\bar a$	-2	0	0	1	0	a	2	0	c	2	ε	$0d$	3
2	f	ε	$(f-1)d$	3	ε	$f\bar c$	4	f	$\bar a$	-2	f	0	1	f	a	2	f	c	2	ε	fd	3
3	bd	b	cd	3	b	$d\bar c$	4	bd	$\bar a$	-2	bd	0	1	bd	a	2	bd	c	2	ε	$(b+1)\bar c\bar d$	5
4	$e\bar c$	ε	$e\bar c\bar d$	5	$e\bar c$	$\bar c$	-2	$e\bar c$	$\bar a$	-2	$e\bar c$	0	1	$e\bar c$	a	2	e	$\bar c c$	-4	e	$\overline{(c-1)}\bar d$	-3
5	$e\bar c\bar d$	$(e-1)d$	cd	3	$(e-1)d$	$d\bar c$	4	$e\bar c\bar d$	$\bar a$	-2	$e\bar c\bar d$	0	1	$e\bar c\bar d$	a	2	$e\bar c$	$\bar d c$	-4	$e\bar c$	$\bar c\bar d$	-3

$$A := \left\{1 \le a \le \frac{q}{2} - 2\right\} \qquad d := \frac{q}{2}$$

$$B := \left\{0 \le b \le \frac{q}{2} - 1\right\} \qquad E := \left\{1 \le e \le \frac{q}{2}\right\}$$

$$c := \frac{q}{2} - 1 \qquad F := \left\{1 \le f \le \frac{q}{2} - 1\right\}$$

TABLE 5. Transducer to compute a minimal expansion in base q from left to right

	In: $\bar{1}$		In: 0		In: 1	
from	Out:	To:	Out:	To:	Out:	To:
0	ε	$0\bar{1}$	0	0	ε	01
01	0	01	0	10	ε	011
10	ε	011	10	0	1	01
011	01	01	ε	0110	10	$0\bar{1}$
0110	01	011	0110	0	10	$0\bar{1}\bar{1}$
$0\bar{1}$	ε	$0\bar{1}\bar{1}$	0	$\bar{1}0$	0	$0\bar{1}$
$\bar{1}0$	$\bar{1}$	$0\bar{1}$	$\bar{1}0$	0	ε	$0\bar{1}\bar{1}$
$0\bar{1}\bar{1}$	$\bar{1}0$	01	ε	$0\bar{1}\bar{1}0$	$0\bar{1}$	$0\bar{1}$
$0\bar{1}\bar{1}0$	$\bar{1}0$	011	$0\bar{1}\bar{1}0$	0	$0\bar{1}$	$0\bar{1}\bar{1}$

Table 6. Transducer to compute a minimal expansion in base 2 from left to right.

from	In: $\bar{1}$	0	1
1	3	4	2
2	—	5	—
3	—	6	—
4	7	4	8
5	3	9	—
6	—	10	2
7	12	11	—
8	—	13	12
9	7	9	—
10	—	10	8
11	7	4	2
12	—	1	—
13	3	4	8

Table 7. Output of the transducer in Table 6.

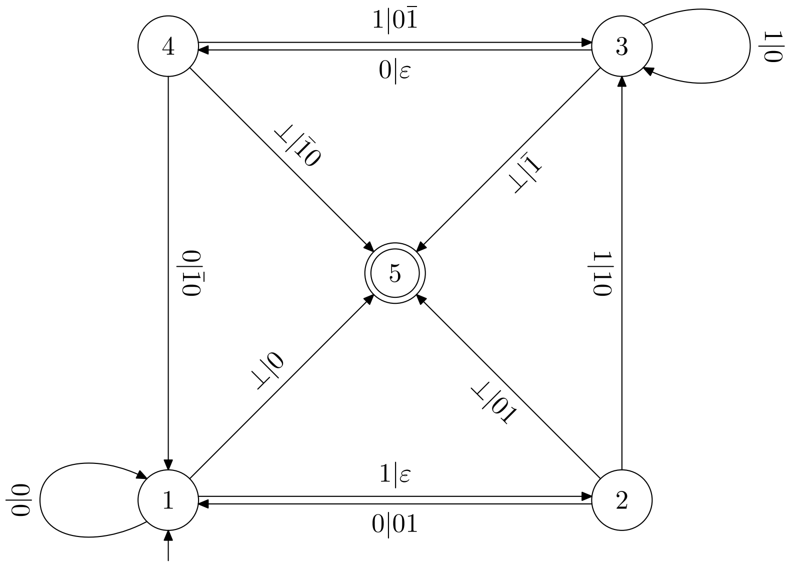

Figure 3. Transducer to compute a minimal expansion in base 2 from left to right starting with the unsigned binary expansion expansion. The symbol $\perp$ denotes the end of the input sequence.

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