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ON PLANARITY AND COLORABILITY OF CIRCULANT GRAPHS

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ABSTRACT. For given postive integers $n, a_1, \dots, a_m$ we consider the undirected circulant graph $G = (V, E)$ with set of vertices $V = \{0, \dots, n-1\}$ and set of edges $E = \{[i, j] : i - j \equiv \pm a_k \pmod n$ for some $1 \le k \le m\}$.

We prove that G is planar if $m = 1$ and non-planar if $m \ge 3$. For $m = 2$ we completely characterize planarity.

It is shown that G is bipartite if and only if there is an l such that 2^l divides $a_1, \dots, a_m$, $2^{l+1} \mid n$, but $2^{l+1} \nmid a_j$ for $1 \le j \le m$.

If $m \le 2$, we also calculate the chromatic number of G.

1. Introduction

Let n, m and $a_1, \dots, a_m$ be positive integers. An undirected graph with set of vertices $V = \{0, \dots, n-1\}$ and set of edges $E = \{[i, i + a_j \bmod n] : 0 \le i \le n-1, 1 \le j \le m\}$ is called a *(symmetric) circulant graph*, since the adjacency matrix of such a graph is usually called a circulant matrix, and it is denoted by $C_n(a_1, \dots, a_m)$.

Since we defined an *undirected* graph, we also have $[i, i - a_j \bmod n] \in E$ for all $0 \le i \le n-1$ and $0 \le j \le m$. If $a_k \not\equiv \pm a_l \pmod n$ for all $1 \le k, l \le m$, $C_n(a_1, \dots, a_m)$ is regular of degree δ, where

$$\delta = \begin{cases} 2m & \text{if } a_j \not\equiv n/2 \pmod n \text{ for all } 1 \le j \le m, \\ 2m-1 & \text{otherwise.} \end{cases}$$

The aim of this paper is to investigate graph theoretic properties of circulant graphs. It is a well-known (and easy-to-prove) fact that a circulant graph $C_n(a_1, \dots, a_m)$ is connected if and only if

$$\gcd(a_1, \dots, a_m, n) = 1; \tag{1}$$

more precisely, it has $\gcd(a_1, \dots, a_m, n)$ isomorphic connected components. We refer to Boesch and Tindell [1] for further results concerning connectivity of circulant graphs.

Hamiltonicity properties of circulant graphs have been studied by Burkard and Sandholzer [3] who prove that a circulant graph is Hamiltonian if and only if it is connected. For the case of directed circulant graphs with two stripes we refer to Yang, Burkard, Çela, and Woeginger [11].

In this paper, we will deal with planarity (Section 4), bipartiteness (Section 2) and the chromatic number (Section 3) of circulant graphs. The first two questions will be fully answered, for the chromatic number we restrict ourselves to the case $m \le 2$.

A related family of graphs are Toeplitz graphs: Let n, m and $1 \le a_1, \dots, a_m < n$ be positive integers. An undirected graph with set of vertices $V = \{0, \dots, n-1\}$ (or $V = \mathbb{N}$) and set of edges $E = \{[i, i + a_j] : i, i + a_j \in V, 1 \le j \le m\}$ is called a finite (or infinite) Toeplitz graph, respectively. It is denoted by $T_n(a_1, \dots, a_m)$ (or $T_\infty(a_1, \dots, a_m)$). It is clear that $C_n(a_1, \dots, a_m) = T_n(a_1, n - a_1, a_2, n - a_2, \dots, a_m, n - a_m)$.

Bipartiteness of Toeplitz graphs can be decided if $V = \mathbb{N}$ (cf. Euler, Le Verge, and Zamfirescu [8]) or if $n < \infty$ and $m = 2$ (cf. Euler [7]). Some necessary and sufficient conditions for bipartiteness for the case $n < \infty$ and $m = 3$ are also given in [7].

Planarity of infinite Toeplitz graphs is decided in Euler [6], where the chromatic number of planar infinite Toeplitz graphs (we remark that planarity implies $m \le 3$) is also determined.

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Similarly, infinite Toeplitz graphs over $V = \mathbb{Z}$ have been studied; we refer to Eggleton, Erdős, and Skilton [5], Walther [10], and Chen, Chang, and Huang [4] and the references therein.

In the remainder of this paper, we will consider circulant graphs $C_n(a_1, \ldots, a_m)$ which we will assume to be *properly given*, i. e., $a_i \not\equiv \pm a_j \pmod{n}$ for $i \neq j$. In view of (1) we will also assume that $d := \gcd(a_1, \ldots, a_m, n) = 1$ since $C_n(a_1, \ldots, a_m)$ is bipartite (planar, k-colorable) if and only if its connected components are; and these connected components are isomorphic to $C_{n/d}(a_1/d, \ldots, a_m/d)$.

We define $a \bmod n$ to be the unique integer $r \in \{0, \ldots, n-1\}$ such that $a \equiv r \pmod{n}$. For a prime number p we will denote the p-adic valuation by v_p, i. e., for any integer k, $v_p(k)$ is defined to be the largest $l \in \mathbb{Z}$ such that $p^l \mid k$. We recall that for all integers k_1, k_2

$$v_p(k_1 + k_2) \geq \min\{v_p(k_1), v_p(k_2)\}, \tag{2a}$$

$$v_p(k_1 + k_2) = \min\{v_p(k_1), v_p(k_2)\} \qquad \text{if } v_p(k_1) \neq v_p(k_2), \tag{2b}$$

$$v_p(k_1 \cdot k_2) = v_p(k_1) + v_p(k_2). \tag{2c}$$

2. Bipartiteness

Bipartite circulant graphs can be characterized as follows:

Theorem 2.1. *Let $G := C_n(a_1, \ldots, a_m)$ be a connected circulant. Then G is bipartite if and only if*

$$a_1, \ldots, a_m \text{ are odd and } n \text{ is even.} \tag{3}$$

Proof. Obviously, G is bipartite if and only if there is no odd cycle, i. e., there are no $x_0, \ldots, x_m \in \mathbb{Z}$ with $\sum_{i=1}^m x_i \equiv 1 \pmod{2}$ and

$$x_0 n + \sum_{i=1}^{m} a_i x_i = 0. \tag{4}$$

Equivalently (defining u by $\sum_{i=1}^m x_i = 2u + 1$ and eliminating x_m from this equation and (4)), there are no $x_0, x_1, \ldots, x_{m-1}, u \in \mathbb{Z}$ such that

$$x_0 n + \sum_{i=1}^{m-1} x_i(a_i - a_m) + 2a_m u = -a_m.$$

We conclude that G is bipartite if and only if

$$d' := \gcd(n, a_1 - a_m, \ldots, a_{m-1} - a_m, 2a_m) \nmid a_m. \tag{5}$$

Let $p \neq 2$ be a prime. From $v_p(2) = 0$ and (2) we obtain

$$\begin{aligned} v_p(d') &= \min\{v_p(n), v_p(a_1 - a_m), \ldots, v_p(a_{m-1} - a_m), v_p(a_m)\} \\ &\geq \min\{v_p(n), v_p(a_1), \ldots, v_p(a_m)\} = v_p(\gcd(n, a_1, \ldots, a_m)) = v_p(1) = 0 \\ &\geq \min\{v_p(n), v_p(a_1 - a_m), v_p(a_m), v_p(a_2 - a_m), v_p(a_m), \ldots, v_p(a_{m-1} - a_m), v_p(a_m)\} \\ &= v_p(d'). \end{aligned}$$

This shows that (5) is equivalent to

$$v_2(d') > v_2(a_m). \tag{6}$$

Assume now that G is bipartite and therefore (6) holds. By definition of d' this implies $v_2(a_i - a_m) \geq v_2(d') > v_2(a_m)$ for $1 \leq i \leq m-1$. If $v_2(a_i) \neq v_2(a_m)$ for some i, (2b) shows that $v_2(a_m) \geq \min\{v_2(a_i), v_2(a_m)\} = v_2(a_i - a_m) > v_2(a_m)$, which is a contradiction. It follows that $v_2(a_1) = v_2(a_2) = \cdots = v_2(a_m)$. In addition, we see $v_2(n) \geq v_2(d') > v_2(a_m)$ which proves (3) since we assumed that G is connected.

Conversely, assume (3). This immediately implies $v_2(a_i - a_m) \geq 1$ for $1 \leq i \leq m-1$ and therefore

$$v_2(d') = \min\{v_2(n), v_2(a_1 - a_m), \ldots, v_2(a_{m-1} - a_m), v_2(a_m) + 1\} \geq 1 > v_2(a_m) = 0,$$

which yields (6). $\square$

3. Chromatic Number

This section is devoted to the calculation of the chromatic number $\chi(G)$ for circulant graphs with $m \leq 2$, i. e., the minimum number of colors needed to color the vertices of G such that adjacent vertices do not have the same color.

Of course, the case $m = 1$ is trivial, however, we state it for the sake of completeness:

Theorem 3.1. *Let $G := C_n(a)$ be a properly given connected circulant graph. Then*

$$\chi(G) = \begin{cases} 2 & n \text{ is even,} \\ 3 & n \text{ is odd.} \end{cases}$$

The proof of the following theorem for $m = 2$ will be the content of this section:

Theorem 3.2. *Let $G := C_n(a, b)$ be a properly given connected circulant graph. Then*

$$\chi(G) = \begin{cases} 2 & \text{if } a \text{ and } b \text{ are odd and } n \text{ is even,} \\ 4 & \text{if } 3 \nmid n,\ n \neq 5, \text{ and } (b \equiv \pm 2a \pmod{n} \text{ or } a \equiv \pm 2b \pmod{n}), \\ 4 & \text{if } n = 13 \text{ and } (b \equiv \pm 5a \pmod{13} \text{ or } a \equiv \pm 5b \pmod{13}), \\ 5 & \text{if } n = 5, \\ 3 & \text{otherwise.} \end{cases} \tag{7}$$

The case $\chi(G) = 2$ is fully characterized by Theorem 2.1. To get an upper bound for $\chi(G)$, we use the following result:

Lemma 3.3 (Brooks [2])**.** *Let G be a graph such that all vertices have degree at most $d > 2$ and such that none of its connected components is a complete graph of order $d + 1$. Then $\chi(G) \leq d$.*

Hence, $C_n(a, b)$ can certainly be colored with 4 colors unless $n = 5$, where the only properly given circulant is $C_5(1, 2) = K_5$ which has chromatic number 5.

Therefore we only have to decide which $C_n(a, b)$ can be colored with 3 colors. This will mostly be done by an explicit construction of a 3-coloring using the colors B (black or blue, as you like), G (green) and R (red) (or sometimes simply $0, 1, 2$).

These colorings will usually be given as elements of the free semigroup generated by $\{B, G, R\}$, for instance, $((BG)^2R)^3BR$ has to be read as $BGBGRBGBGRBGBGRBR$. We define the rotation by l of such a word as follows:

$$x_0x_1 \ldots x_{r-1} \operatorname{rot} l := x_{r-l}x_{r-(l-1)} \ldots x_{r-1}x_0x_1 \ldots x_{r-l-1}.$$

We will use the following isomorphism several times:

Lemma 3.4. *Let $C_n(a, b)$ be a properly given circulant and* $\gcd(a, n) = 1$. *Then the graph $C_n(a, b)$ is isomorphic to the graph $C_n(1, a^{-1}b \bmod n)$.*

Proof. Trivial. □

3.1. Special case: $C_n(1, a)$ with $\gcd(a, n) = 1$.

In view of Lemma 3.4, we will first consider the special case of $G = C_n(1, a)$ with $\gcd(a, n) = 1$. If n is even, a has to be odd, and $\chi(G) = 2$ by Theorem 2.1. In the remainder of this subsection, we will focus on the case of odd n. Since $C_n(1, a) = C_n(1, n - a)$ we may restrict ourselves to the case $2 \leq a \leq (n - 1)/2$.

Lemma 3.5. *Let n be odd and $a \in \{2, (n - 1)/2\}$. Then $G = C_n(1, a)$ is 3-colorable if and only if $3 \mid n$.*

Proof. Consider first $a = 2$ and let $c : V \to \{0, 1, 2\}$ be a 3-coloring of G. Without loss of generality, we have $c(0) = 0$ and $c(1) = 1$. We claim that for $0 \leq k < n$, $c(k) = k \bmod 3$. Assume that the claim is true for $0 \leq k - 1, k < n - 1$. Since $[k, k + 1] \in E$ and $[k - 1, k + 1] \in E$, we get $c(k+1) \neq c(k) = k \bmod 3$ and $c(k+1) \neq c(k-1) = k-1 \bmod 3$, which implies $c(k+1) = k+1 \bmod 3$ and proves the claim. Since $[n - 2, 0] \in E$ and $[n - 1, 0] \in E$, we get $0 \not\equiv n - 2 \pmod{3}$ and $0 \not\equiv n - 1 \pmod{3}$ and consequently $3 \mid n$.

Conversely, if $3 \mid n$, $c(i) := i \bmod 3$, $0 \leq i < n$, defines a valid 3-coloring of G.

Consider now $a = (n-1)/2$. Since $\gcd(a, n) = \gcd((n-1)/2, (n+1)/2) = \gcd((n-1)/2, 1) = 1$ and since $2a \equiv -1 \pmod{n}$, $C_n(1, a)$ is isomorphic to $C_n(1, -2) = C_n(1, 2)$ by Lemma 3.4. This graph has just been considered above. □

Lemma 3.6. *Let n be odd and*

$$\max\left\{2, \frac{n-3}{3}\right\} < a \leq \frac{n-3}{2}. \tag{8}$$

Then $C_n(1, a)$ is 3-colorable if and only if $(n, a) \neq (13, 5)$.

Proof. Assume first $(n, a) \neq (13, 5)$.

Write $n = 2(a+1) + 2s + 1$ for some integer s with $0 \leq s < a/2$—this is possible by (8)—and $a = (2s+3)q + t$ for some integers q and t with $0 \leq t \leq 2s+2$.

If t is odd, we use the coloring $X := ((BG)^{s+1}R)^q(BG)^{(t-1)/2}B((GR)^{s+1}B)^{q+1}(RB)^{(t-1)/2}R$. It is easy to see that the length of X is indeed n and that this coloring handles edges $[k, k+1]$ correctly; in order to verify edges $[k, k+a]$, we calculate $X \operatorname{rot} a$ and see from

$$X = ((BG)^{s+1}R)^q(BG)^{(t-1)/2}B((GR)^{s+1}B)^q(GR)^{(t-1)/2}G(RG)^{s-((t-1)/2)}RB(RB)^{(t-1)/2}R,$$

$$X \operatorname{rot} a = ((GR)^{s+1}B)^q(RB)^{(t-1)/2}R((BG)^{s+1}R)^q(BG)^{(t-1)/2}B(GR)^{s-((t-1)/2)}GR(GR)^{(t-1)/2}B,$$

that the coloring is indeed valid. We will give many colorings of this shape, but we will restrict ourselves to giving the coloring X itself (mostly in tabular form) and we will omit the (tedious) routine verification as demonstrated in this example only.

The cases for even t are discussed in Table 3.1.

$2 \leq t \leq 2s, 2 \mid t$	$((BG)^{s+1}R)^q(BG)^{t/2}((RB)^{s+1}G)^q(RB)^{t/2}(GR)^{s+1-(t/2)}(BR)^{(t/2)-1}BGR,$
$t = 0$	$((BG)^{s+1}R)^q((GR)^{s+1}B)^q(RB)^{s+1}G,$
$t = 2s+2, s \geq 1$	$((BG)^{s+1}R)^{2q+2}(BG)^sR,$
$t = 2s+2, s = 0,$ $n \geq 19$	$RBGBGRGR(BGR)^{((n-1)/6)-2}GRBR(BGR)^{((n-1)/6)-2}B.$

TABLE 3.1. Lemma 3.6, t even

Let us now turn to the very special case $n = 13$ and $a = 5$. Assume that there is a 3-coloring $c : \{0, \dots, 12\} \to \{0, 1, 2\}$ of $C_{13}(1, 5)$. Since $3 \nmid n$ there must be some i such that $c((i-1) \bmod 13) = c((i+1) \bmod 13)$. Without loss of generality, we have $c(0) = 0, c(1) = 1, c(2) = 0$.

If we assume that $c(12) = c(3) = 1$, we get a contradiction as shown in Table 3.2, therefore $c(3) = 2$ or $c(12) = 2$. By symmetry, we may assume $c(3) = 2$, which is lead to a contradiction in Table 3.3.

$[0, 8], [3, 8] \in E \Rightarrow$	$c(8) = 2,$
$[8, 9], [1, 9] \in E \Rightarrow$	$c(9) = 0,$
$[2, 7], [7, 8] \in E \Rightarrow$	$c(7) = 1,$
$[7, 12] \in E$	$\Rightarrow$ Contradiction.

TABLE 3.2. Lemma 3.6, $c(0) = c(2) = 0, c(1) = c(3) = c(12) = 1$

□

Lemma 3.7. *Let n be odd and*

$$3 \leq a \leq \frac{n-3}{3}. \tag{9}$$

Then $C_n(1, a)$ is 3-colorable.

$$\begin{array}{lrr}
[0,8],[3,8]\in E & \Rightarrow & c(8)=1,\\
[7,8],[2,7]\in E & \Rightarrow & c(7)=2,\\
[7,12],[12,0]\in E & \Rightarrow & c(12)=1,\\
[4,12],[3,4]\in E & \Rightarrow & c(4)=0,\\
[4,9],[8,9]\in E & \Rightarrow & c(9)=2,\\
[2,10],[9,10]\in E & \Rightarrow & c(10)=1,\\
[5,10],[4,5]\in E & \Rightarrow & c(5)=2,\\
[1,6],[5,6]\in E & \Rightarrow & c(6)=0,\\
[3,11],[6,11]\in E & \Rightarrow & c(11)=1,\\
[10,11]\in E & \Rightarrow & \text{Contradiction.}
\end{array}$$

TABLE 3.3. Lemma 3.6, $c(0) = c(2) = 0, c(1) = 1, c(3) = 2$

Proof. Let $r := \lfloor n/(a+1)\rfloor$. Condition (9) implies that $r \geq 3$.

Consider first the case $2 \mid a$ and $2 \nmid r$. We can write $n = r(a+1) + 2s$ for some $0 \leq s \leq a/2$. Table 3.4 proves the lemma in this case.

$s \leq (a/2) - 1$	$((BG)^{a/2}R)^{r-1}(BR)^s B(GR)^{a/2}$,
$s = a/2$	$((BG)^{a/2}R)^{r-1}B(GR)^{(a/2)-1}BG(RB)^{(a/2)-1}GR$.

TABLE 3.4. Lemma 3.7, $2 \mid a$, $2 \nmid r$, $n = r(a+1) + 2s$

a odd	$(BG)^{(n-2a-1)/2}B(RB)^{(a-1)/2}R(GR)^{(a-1)/2}G$,
a even, r even	$((BG)^{a/2}R)^{r-2}B(RB)^s(GR)^{a/2}BG(RB)^{(a/2)-1}GR$.

TABLE 3.5. Lemma 3.7, $2 \mid r(a+1)$, $n = r(a+1) + 2s + 1$

Otherwise, $2 \mid r(a+1)$, and we can write $n = r(a+1) + 2s + 1$ for some $0 \leq s \leq a/2$. The resulting cases are discussed in Table 3.5. $\square$

This concludes the subsection on the special case $C_n(1, a)$ with $\gcd(n, a) = 1$.

3.2. **Associated Hermite Normal Form.** We will now describe how to associate a Hermite Normal Form (HNF) to a circulant graph. This will enable us to describe a graph which is isomorphic to the given circulant and which we will be able to color. Although we will only use these results in the case $m = 2$, we will first work with arbitrary m, because this may be of independent interest and seems to describe the structure in more detail.

We will use the notion of a HNF as it is defined in Schrijver [9, Chapter 4]: A rational matrix A is in HNF if it has the form $A = \begin{pmatrix} B & 0 \end{pmatrix}$, where B is a lower triangular, nonnegative, nonsingular matrix, in which each row has a unique maximum entry, which is located on the main diagonal of B.

Theorem 3.8. *Let $n > 0, a_1, \ldots, a_m$ be integers.*

Then there is a unique matrix $X \in \mathbb{Z}^{m\times m}$ with entries x_{ij} such that

(10a) *X is in HNF,*
(10b) $(a_1, \ldots, a_m) \cdot X \in n\mathbb{Z}^m$,
(10c) $x_{ii} = \gcd(a_{i+1}, \ldots, a_m, n)/\gcd(a_i, a_{i+1}, \ldots, a_m, n)$ *for* $1 \leq i \leq m$.

Furthermore, this matrix has the following properties:

(10d) *If* $y_1a_1 + \cdots + y_ma_m \equiv 0 \pmod{n}$ *for some integers* $y_1, \ldots, y_m$ *and* $j := \min\{i : n \nmid y_i\}$, *then* $x_{jj} \mid y_j$,

(10e) $\operatorname{range}_{\mathbb{Z}}(X) = \{u \in \mathbb{Z}^m : (a_1, \ldots, a_m) \cdot u \equiv 0 \pmod{n}\}$.

Proof. For $1 \le j \le m$, we define $d_j := \gcd(a_j, \ldots, a_m, n)$. We denote the columns of X by $x_1, \ldots, x_m$.

First, we prove that assuming (10a) and (10b), the three conditions (10c), (10d), and (10e) are equivalent.

We show that (10c) implies (10d): Since

$$y_ja_j = y_0n - y_{j+1}a_{j+1} - \cdots - y_ma_m$$

for some integer y_0, it follows that $d_{j+1} \mid y_ja_j$ which leads to $x_{jj} = (d_{j+1}/d_j) \mid y_j$.

Next, let X be a matrix with properties (10a), (10b), and (10d) and let $\Lambda := \{u \in \mathbb{Z}^m : (a_1, \ldots, a_m) \cdot u \equiv 0 \pmod{n}\}$. Property (10b) implies that $\operatorname{range}_{\mathbb{Z}}(X) \subseteq \Lambda$.

Assume that this is a proper inclusion, and choose some $u \in \Lambda \setminus \operatorname{range}_{\mathbb{Z}}(X)$ such that $j := \min\{i : u_i \neq 0\}$ is maximal. By (10d), $q := u_j/x_{jj}$ is an integer. We define $v := u - qx_j$ and note that $v \in \Lambda$ by (10b) and $v_1 = \cdots = v_j = 0$. Therefore $v \in \operatorname{range}_{\mathbb{Z}}(X)$ by the choice of u. However, this implies $u \in \operatorname{range}_{\mathbb{Z}}(X)$, a contradiction, which proves (10e).

Finally, assume that X fulfills (10a), (10b), and (10e) and let $1 \le i \le m$. As in the proof of (10d) above, we see that $(d_{i+1}/d_i) \mid x_{ii}$ since $(a_1, \ldots, a_m) \cdot x_i \equiv 0 \pmod{n}$. Conversely, let $y_i := d_{i+1}/d_i$. It follows that $d_{i+1} \mid a_iy_i$. Therefore, there exist integers $y_{i+1}, \ldots, y_m$ such that $(0, \ldots, 0, y_i, y_{i+1}, \ldots, y_m) \in \Lambda$. Condition (10e) and the fact that X is lower triangular by (10a) yield $x_{ii} \mid y_i = d_{i+1}/d_i$, which proves (10c) and completes the proof of the equivalence of (10c), (10d), and (10e).

Since there is only one HNF with $\mathbb{Z}$-range Λ by Schrijver [9, Theorem 4.2], conditions (10) describe a unique matrix.

Finally, we have to prove existence. Let $Q = (q_1, \ldots, q_m) \in \mathbb{Z}^{(m+1)\times m}$ be a system of fundamental solutions to the linear Diophantine equation

$$a_1x_1 + \cdots + a_mx_m + nx_{m+1} = 0,$$

which can be calculated by Schrijver [9, Corollary 5.3c]. The first m rows of Q are denoted by Q'. It is easy to see that $\operatorname{range}_{\mathbb{Z}}(Q') = \Lambda$ which implies that Q' has full row rank. Denote by X the HNF of Q' (which can be obtained as in Schrijver [9, Theorem 5.3]). We immediately see that X fulfills (10a), (10b), and (10e). □

The matrix X described in Theorem 3.8 will be called the *Hermite Normal Form associated with* $n, (a_1, \ldots, a_m)$. The lattice Λ described in (10e) will be called the *lattice associated with* $n, (a_1, \ldots, a_m)$.

Proposition 3.9. *Let* $G = C_n(a_1, \ldots, a_m)$ *be a properly given connected circulant and* X *and* Λ *the HNF and the lattice associated with* $n, (a_1, \ldots, a_m)$.

Let $V' := \mathbb{Z}^m/\Lambda$ *and*

$$E' := \bigcup_{u+\Lambda \in \mathbb{Z}^m/\Lambda} \bigcup_{i=1}^{m} \{[u + \Lambda, u + e_i + \Lambda]\},$$

where e_i *denotes the* i*-th unit vector,* $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)^t$.

Then the graphs G *and* $G' := (V', E')$ *are isomorphic.*

Note that (10d) and (10e) imply that a complete system of representatives for $\mathbb{Z}^m/\Lambda$ is given by $V'' := \{(s_1, \ldots, s_m) \in \mathbb{Z}^m : 0 \le s_i < x_{ii}\}$.

Proof. Define $\varphi : \mathbb{Z}^m \to \mathbb{Z}/n\mathbb{Z}$ by $(s_1, \ldots, s_m) \mapsto s_1a_1 + \cdots + s_ma_m + n\mathbb{Z}$. Clearly, this is a group homomorphism. As $\gcd(a_1, \ldots, a_m, n) = 1$, it is surjective. By (10e) its kernel is Λ, therefore it induces a group isomorphism $\bar\varphi$ from V' to V.

We note that E' is well defined and that $[u+\Lambda, v+\Lambda] \in E'$ if and only if $v-u \equiv \pm e_i \pmod{\Lambda}$ for some $1 \le i \le n$. This is equivalent to $\bar\varphi(v+\Lambda) - \bar\varphi(u+\Lambda) = \pm a_i + n\mathbb{Z}$, i. e., $[\bar\varphi(u+\Lambda), \bar\varphi(v+\Lambda)] \in E$ which proves that $\bar\varphi$ is a graph isomorphism. □

In the case $m = 2$ the consequences of Proposition 3.9 are as follows:

Corollary 3.10. *Let $C_n(a, b)$ be a properly given connected circulant and X the HNF associated with $n, (a, b)$. Then $x_{11} = \gcd(b, n)$ and $x_{22} = n/\gcd(b, n) > 1$.*

$C_n(a, b)$ can be 3-colored if and only if there are $w_0, \ldots, w_{x_{11}} \in \{B, G, R\}^{x_{22}}$ such that for all $0 \le i \le x_{11}$ and $0 \le j < x_{22}$ the following equations hold:

$$w_{x_{11}} = w_0 \operatorname{rot} x_{21}, \tag{11a}$$

$$w_{ij} \ne w_{i,(j+1) \bmod x_{22}}, \tag{11b}$$

$$w_{ij} \ne w_{(i+1),j}, \tag{11c}$$

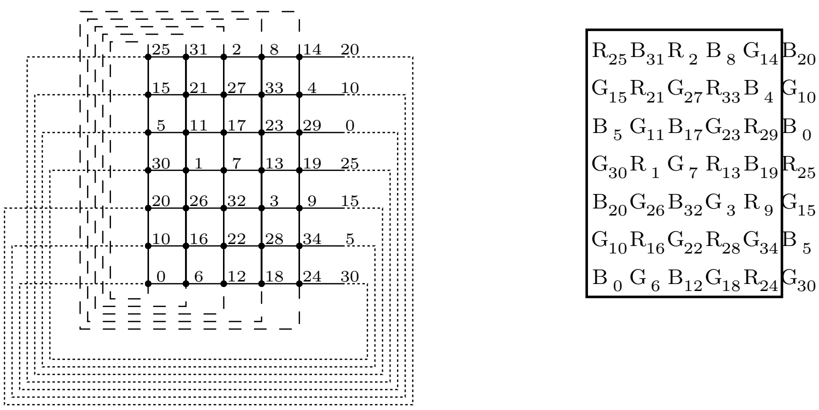
where the components of the w_i are counted from 0, i. e., $w_i = (w_{i0}, \ldots, w_{i,x_{22}-1})$.

Proof. The formulæ for x_{11} and x_{22} follow from (10c), $\gcd(a, b, n) = 1$, and $n \nmid b$.

Let $V'' = \{0, \ldots, x_{11} - 1\} \times \{0, \ldots, x_{22} - 1\}$. By Proposition 3.9, $C_n(a, b)$ can be 3-colored if and only if there is a mapping $w : V'' \to \{R, G, B\}$ such that $w(i, j) \ne w(k, l)$ whenever $(k, l) \equiv (i + 1, j) \pmod{\Lambda}$ or $(k, l) \equiv (i, j + 1) \pmod{\Lambda}$.

By (10d) and the definition of V'', $(k, l) \equiv (i, j+1) \pmod{\Lambda}$ is equivalent to $i = k$ and $l \equiv j+1 \pmod{x_{22}}$, which corresponds to (11b).

Similarly, $(k, l) \equiv (i + 1, j) \pmod{\Lambda}$ is equivalent to ($k = i + 1$ and $l = j$) or to ($k = 0$ and $i = x_{11} - 1$). The first case corresponds to (11c). In the second case, we have $(0, l) \equiv (x_{11}, j) \pmod{\Lambda}$ or equivalently $(0, l + x_{21} - j) \pmod{\Lambda}$, which is equivalent to $l + x_{21} \equiv j \pmod{x_{22}}$. It can easily be checked that this is handled by the extension of the coloring to $w_{x_{11}}$ as in (11a) and the application of (11c). $\square$



(a) $C_{35}(6, 10)$ (b) $C_{35}(6, 10)$ colored

FIGURE 3.1. Example 3.11

Example 3.11. $C_{35}(6, 10)$. We get $x_{11} = \gcd(10, 35) = 5$, $x_{22} = n/x_{11} = 7$. We have to choose x_{21} such that $5 \cdot 6 + x_{21} \cdot 10 \equiv 0 \pmod{35}$ and see that $x_{21} = 4$ is a solution to this congruence. By (10a), (10b), and (10c), this is indeed the associated HNF.

We draw the graph putting vertex $ia + jb$ in position (i, j) (cf. Figure 3.1(a)). In the sixth column from the left, the first column is repeated (rotated by x_{21}); so the dotted lines do not have to be followed anymore.

A coloring is given in Figure 3.1(b), namely:

$$\begin{aligned} w_0 = w_2 &= (BG)^3R, \\ w_1 = w_3 &= (GR)^3B, \\ w_4 &= RG(RB)^2G, \\ w_5 = w_0 \operatorname{rot} 4 &= GBGRBGB. \end{aligned}$$

To conclude, we mention the following result which relates different associated HNFs to the same circulant (note that the HNF is only unique when $n, (a,b)$ are fixed).

Lemma 3.12. *Let $C_n(a,b)$ be a properly given connected circulant and X the HNF associated to $n, (a,b)$. Then the HNF associated to $n, (a, n-b)$ is*

$$X' = \begin{pmatrix} x_{11} & 0 \\ -x_{21} \bmod x_{22} & x_{22} \end{pmatrix}.$$

Proof. X' satisfies (10a), (10b), (10c). □

3.3. **3-Colorability of General $C_n(a,b)$.** In this subsection, we complete the proof of Theorem 3.2. Let $C_n(a,b)$ be a properly given connected circulant. We may assume without loss of generality that $\gcd(a,n) \le \gcd(b,n)$.

Proposition 3.13. *Let $\gcd(a,n) = \gcd(b,n) = 1$. Then $C_n(a,b)$ is 3-colorable except when*

$$6 \nmid n \text{ and } (b \equiv \pm 2a \pmod n \text{ or } a \equiv \pm 2b \pmod n)$$

or

$$n = 13 \text{ and } (b \equiv \pm 5a \pmod{13} \text{ or } a \equiv \pm 5b \pmod{13}).$$

Proof. Since $C_n(a,b)$ is isomorphic to $C_n(1, a^{-1}b \bmod n)$ by Lemma 3.4, this proposition is a consequence of the results of Subsection 3.1. □

Lemma 3.14. *Let $x_{11} = 2$ and $x_{21} = 1$. Then $C_n(a,b)$ is 3-colorable if and only if $3 \mid x_{22}$.*

Proof. Let $w_0, w_1, w_2 = w_0 \operatorname{rot} 1$ be a coloring according to Corollary 3.10. Instead of the colors B, G, R we will just write $0, 1, 2$. Without loss of generality we have $w_{00} = 0$ and $w_{10} = 1$. We claim that $w_{k,l} = k + l \bmod 3$ for $k = 0, 1, 2$ and $0 \le l \le x_{22} - 1$. Assume that this is true for some $(0, l)$ and $(1, l)$ with $0 \le l < x_{22} - 1$. Since $w_{1,l+1} \ne w_{2,l+1} = w_{0,l} = l \bmod 3$ and $w_{1,l+1} \ne w_{1,l} = l + 1 \bmod 3$, we get $w_{1,l+1} = l + 2 \bmod 3$. This implies that $w_{0,l+1} = l + 1 \bmod 3$, and the claim is proven for $(0, l+1), (1, l+1)$. From that, it follows for $k = 2$ also.

As $w_{0,x_{22}-1} \ne w_{0,0} = 0$ and $w_{0,x_{22}-1} = w_{2,0} \ne w_{0,1} = 1$ we get $x_{22} - 1 \bmod 3 = 2$, i. e., $3 \mid x_{22}$.

Conversely, if $3 \mid x_{22}$, the above coloring is indeed valid. □

Lemma 3.15. *Let $x_{11} = 2$ and $2 \nmid x_{21} \ge 3$. Then $C_n(a,b)$ is not 3-colorable if and only if $x_{21} = x_{22} - 1$ and $3 \nmid x_{22}$.*

Proof. Let $r := \lfloor x_{22}/x_{21} \rfloor$ and $u \ge 1$ such that $x_{21} = 2u + 1$.

If $x_{22} - rx_{21}$ is even, say $2s$ for some $0 \le s \le u$, then

$$\begin{aligned} w_0 &= ((BG)^u R)^r (BR)^s, \\ w_1 &= ((GR)^u B)^r (GB)^s \end{aligned}$$

gives a valid 3-coloring.

Therefore, we may assume $x_{22} = rx_{21} + 2s + 1$ for some $0 \le s < u$. If $r \ge 2$, the result can be found in Table 3.6.

Finally, we have to deal with the case $x_{22} = x_{21} + 2s + 1$ with $0 \le s < u$. Let $x'_{21} := x_{22} - x_{21} = 2s + 1 = -x_{21} \bmod x_{22}$. Then we may switch to $C_n(a, n-b)$ according to Lemma 3.12 with the new HNF $\left(\begin{smallmatrix} 2 & 0 \\ x'_{21} & x_{22} \end{smallmatrix}\right)$.

Obviously, $2 \nmid x'_{21}$. If $s \ge 1$, then the new HNF corresponds to those cases of Lemma 3.15 which have already been proven to be 3-colorable, since $x'_{21} < x_{22}/2$.

$1 \le s < u$	$w_0 = ((BG)^u R)^{r-1}(BG)^s R(GR)^{u-s}(BG)^s R,$
	$w_1 = ((RB)^u G)^{r-1}(RB)^s G(BG)^{u-s}(RB)^s G,$
$s = 0$	$w_0 = ((BG)^u R)^{r-1}(BR)^u BG,$
	$w_1 = ((GR)^u B)^{r-1}(GB)^u GR.$

TABLE 3.6. Lemma 3.15, $x_{22} = rx_{21} + 2s + 1$, $r \ge 2$

If $s = 0$, the new HNF corresponds to Lemma 3.14 which gives the exceptional cases in the statement of Lemma 3.15. □

Lemma 3.16. *Let $x_{11} = 2$ and $2 \mid x_{21}$. Then $C_n(a, b)$ is not 3-colorable if and only if $x_{21} = x_{22} - 1$ and $3 \nmid x_{22}$.*

Proof. If $2 \mid x_{22}$, a valid coloring is given by $w_0 = (BG)^{x_{22}/2}$, $w_1 = (GB)^{x_{22}/2}$, so we may assume $2 \nmid x_{22}$. If $x_{21} = 0$, a valid 3-coloring is given by $w_0 := (BG)^s R$ and $w_1 := (RB)^s G$, where $x_{22} = 2s + 1$.

Finally, if $x_{21} > 0$, we have $x'_{21} := x_{22} - x_{21} = -x_{21} \bmod x_{22}$ and observe that $2 \nmid x'_{21}$. Using Lemma 3.12 we can deduce Lemma 3.16 from Lemma 3.14 and Lemma 3.15. □

We sum up the results for $x_{11} = 2$ in the following proposition:

Proposition 3.17. *Let $x_{11} = 2$. Then $C_n(a, b)$ is not 3-colorable if and only if $3 \nmid x_{22}$ and ($x_{21} = 1$ or $x_{21} = x_{22} - 1$).*

Proof. This is a consequence of Lemma 3.14, Lemma 3.15, and Lemma 3.16. □

Lemma 3.18. *Let $x_{11} \ge 3$ and $2 \mid x_{22}$. Then $C_n(a, b)$ is 3-colorable.*

Proof. Write $x_{22} = 2r$. If $2 \mid x_{11} + x_{21}$, then $w_{2u} := (BG)^r$ and $w_{2u+1} := (GB)^r$ for $u \ge 0$ is a valid coloring.

Otherwise $w_0 := (BG)^r$, $w_1 := (RB)^r$, $w_{2u} := (GR)^r$, $w_{2u+1} := (RG)^r$ for $u \ge 1$ is a valid coloring. □

Lemma 3.19. *Let $2 \nmid x_{11} \ge 3$ and $2 \nmid x_{22}$. Then $C_n(a, b)$ is 3-colorable.*

Proof. If x_{21} is even, we write $x_{21} := 2u$, $x_{22} := 2r + 1$, and $x_{11} := 2s + 1$. Valid colorings are given in Table 3.7.

$u \ge 1$	$w_{2t} = (BG)^r R$	for $0 \le t < s$,
	$w_{2t+1} = (GR)^r B$	for $0 \le t < s$,
	$w_{2s} = (RG)^{u-1}(RB)^{r-u+1} G,$	
$u = 0$	$w_{2t} = (BG)^r R$	for $0 \le t < s$,
	$w_{2t+1} = (GR)^r B$	for $0 \le t < s$,
	$w_{2s} = (RB)^r G.$	

TABLE 3.7. Lemma 3.19, $2 \mid x_{21}$.

If $2 \nmid x_{21}$, then we can use Lemma 3.12 to prove the result. □

Lemma 3.20. *Let $2 \mid x_{11} \ge 4$ and $2 \nmid x_{22}$. Then $C_n(a, b)$ is 3-colorable.*

Proof. By Lemma 3.12 we may assume $x_{21} \le x_{22}/2$. If $3 \mid x_{22}$ or $x_{21} > 1$ we consider $X'' := \left(\begin{smallmatrix} 2 & 0 \\ x_{21} & x_{22} \end{smallmatrix}\right)$. By Proposition 3.17 there are w_0'' and w_1'' such that the circulant defined by X'' can be 3-colored. Defining $w_{2u} := w_0''$ and $w_{2u+1} := w_1''$ for $0 \le u < x_{11}/2$ we obtain a valid coloring for $C_n(a,b)$.

We have to consider the remaining case $3 \nmid x_{22}$ and $x_{21} = 1$. We write $x_{22} := 2s+1$. A valid coloring is given by

$$\begin{aligned} w_{2t} &:= (BG)^s R, && 0 \le t < x_{11}/2 - 1, \\ w_{2t+1} &:= (GR)^s B, && 0 \le t < x_{11}/2 - 1, \\ w_{x_{11}-2} &:= (RB)^s G, \\ w_{x_{11}-1} &:= (BG)^s R. \end{aligned}$$

□

We can summarize the case $x_{11} \ge 3$ as follows:

Proposition 3.21. *Let $x_{11} \ge 3$. Then $C_n(a,b)$ is 3-colorable.*

From Propositions 3.13, 3.17, and 3.21 we can easily derive Theorem 3.2.

4. Planarity

In this section, we will prove the following theorem, which characterizes all planar circulant graphs.

Theorem 4.1. *Let $G = C_n(a_1, \dots, a_m)$ be a connected properly given circulant. G is planar if and only if one of the following conditions holds:*

(12a) $m = 1$.
(12b) $m = 2$, $a_i \equiv \pm 2a_j \pmod{n}$, *and* $2 \mid n$, *where* $(i,j) = (1,2)$ *or* $(i,j) = (2,1)$.
(12c) $m = 2$, $a_i = n/2$, *and* $2 \mid a_j$, *where* $(i,j) = (1,2)$ *or* $(i,j) = (2,1)$.

We note that in the case $m = 1$, G is clearly planar. We will now discuss the case $m = 2$ in several subcases, and finally we will turn to the case $m \ge 3$.

4.1. Planarity of Circulant Graphs with $m = 2$.

Let $G = C_n(a,b)$ be a properly given connected circulant. Without loss of generality, we assume $\gcd(b,n) \le \gcd(a,n)$. We consider the associated HNF X. By applying Lemma 3.12 if necessary we may also assume $x_{21} \le x_{22}/2$. Since $x_{11}a + x_{21}b \equiv 0 \pmod{n}$ we get $\gcd(a,n) \mid x_{21}$ which implies $x_{21} = 0$ or $x_{11} = \gcd(b,n) \le \gcd(a,n) \le x_{21}$.

Lemma 4.2. *Let $x_{11} = 1$ and $x_{21} = 2$. Then $C_n(a,b)$ is planar if and only if $2 \mid x_{22}$.*

Proof. We note that the assumptions imply $a + 2b \equiv 0 \pmod{n}$.

Assume first that $2 \nmid x_{22}$ and write $x_{22} = 2r+1$. The case $r = 0$ cannot happen since it would imply $n \mid b$. If $r = 1$ then $n \mid 3b$ which yields $a \equiv \pm b \equiv \pm n/3 \pmod{n}$ and G was not properly given. If $r = 2$ then $n = 5$ and it is easily seen that $G = K_5$ and therefore G is non-planar.

We assume $x_{22} = 2r+1$ where $r \ge 3$. We consider the subgraph $G' = (V, E')$ of G described in Table 4.1, which is to be read as follows: $[u,v] \in E'$ if and only if $u \leftrightarrow v$ is listed somewhere in Table 4.1. All congruences in Table 4.1 are meant to be modulo n.

We remark that vertices $-(r-3)b, -(r-4)b, \dots, -b, 0, b, \dots, (r-4)b, (r-3)b$ have degree 2 in G', therefore after removing them we end up with $K_{3,3}$, which proves non-planarity in this case.

Finally, we have to deal with $x_{22} = 2r$ for some r. Figure 4.1 gives a planar embedding for this case.

□

Lemma 4.3. *Let $x_{11} = 1$, $x_{21} \ge 3$. Then $C_n(a,b)$ is non-planar.*

Proof. By assumption we have $a \equiv -x_{21}b \pmod{n}$. Since $3 \le x_{21} \le x_{22}/2$ we get $x_{22} \ge 6$ which implies that $\{0, b, 2b, x_{21}b, (x_{21}+1)b, (x_{21}+2)b\}$ are pairwise incongruent modulo n. We consider the subgraph G' described in Table 4.2. Vertices $3b, \dots, (x_{21}-1)b$ and $(x_{21}+2)b, \dots, (x_{22}-1)b$ have

$rb \leftrightarrow (r+1)b \equiv -rb,$
$rb \leftrightarrow rb - a \equiv (r+2)b \equiv -(r-1)b,$
$rb \leftrightarrow rb + a \equiv (r-2)b,$
$(r-1)b \leftrightarrow (r-1)b - a \equiv -rb,$
$(r-1)b \leftrightarrow (r-1)b + a \equiv (r-3)b \leftrightarrow (r-5)b \leftrightarrow \ldots \leftrightarrow (r-1-2(r-1))b \equiv -(r-1)b,$
$(r-1)b \leftrightarrow (r-2)b,$
$-(r-2)b \leftrightarrow -(r-2)b + a \equiv -rb,$
$-(r-2)b \leftrightarrow -(r-1)b,$
$-(r-2)b \equiv (r-2-2(r-2))b \leftrightarrow (r-2-2(r-3))b \leftrightarrow \ldots \leftrightarrow (r-4)b \leftrightarrow (r-2)b.$

TABLE 4.1. Lemma 4.2, $x_{22} = 2r+1$, $r \geq 3$

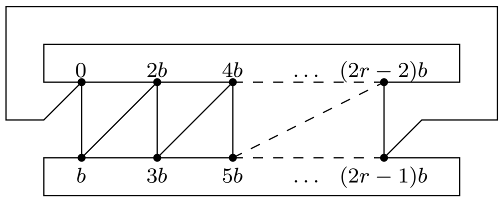


FIGURE 4.1. Lemma 4.2, $x_{22} = 2r$

$b \leftrightarrow 0,$
$b \leftrightarrow 2b,$
$b \leftrightarrow b - a \equiv (x_{21}+1)b,$
$x_{21}b \equiv -a \leftrightarrow 0,$
$x_{21}b \leftrightarrow (x_{21}-1)b \leftrightarrow \ldots \leftrightarrow 3b \leftrightarrow 2b,$
$x_{21}b \leftrightarrow (x_{21}+1)b,$
$(x_{21}+2)b \leftrightarrow (x_{21}+3)b \leftrightarrow \ldots \leftrightarrow (x_{22}-1)b \leftrightarrow 0,$
$(x_{21}+2)b \leftrightarrow (x_{21}+2)b + a \equiv 2b,$
$(x_{21}+2)b \leftrightarrow (x_{21}+1)b.$

TABLE 4.2. Lemma 4.3

degree 2 and can be removed, and we end up with an instance of $K_{3,3}$. This proves non-planarity in this case. □

Lemma 4.4. *Let $x_{21} = 0$. Then $C_n(a, b)$ is planar if and only if $x_{11} = 2$.*

Proof. We note that the assumption $x_{21} = 0$ implies $x_{11}a \equiv 0 \pmod{n}$ which yields $x_{11} \geq 2$.

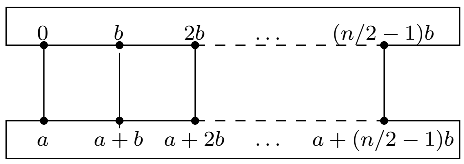


FIGURE 4.2. Lemma 4.4, $x_{11} = 2$

If $x_{11} = 2$ we get $a = n/2$, $x_{22} = n/x_{11} = n/2$, and a planar embedding is given in Figure 4.2.

$$
\begin{aligned}
&a \leftrightarrow 0 \leftrightarrow b,\\
&a \leftrightarrow a+b,\\
&a \leftrightarrow a+(x_{22}-1)b \leftrightarrow a+(x_{22}-2)b \leftrightarrow \ldots \leftrightarrow a+3b \leftrightarrow a+2b,\\
&a \leftrightarrow 2a \leftrightarrow 2a+b,\\
&b \leftrightarrow a+b,\\
&b \leftrightarrow 2b \leftrightarrow a+2b,\\
&b \leftrightarrow (x_{11}-1)a+b \leftrightarrow (x_{11}-2)a+b \leftrightarrow \ldots \leftrightarrow 3a+b \leftrightarrow 2a+b,\\
&a+b \leftrightarrow a+2b,\\
&a+b \leftrightarrow 2a+b,\\
&a+2b \leftrightarrow 2a+2b \leftrightarrow 2a+b.
\end{aligned}
$$

TABLE 4.3. Lemma 4.4, $x_{11} \geq 3$

Let $x_{11} \geq 3$. Consider the subgraph G' constructed in Table 4.3. We remark that the vertices $0, 2a, 2b, 2a+2b$ as well as $a+3b, \ldots, a+(x_{22}-1)b$ and $3a+b, \ldots, (x_{11}-1)a+b$ have degree 2 in G' and can be removed, yielding an instance of K_5. □

Lemma 4.5. *Let $x_{11} \geq 2$ and $x_{21} \neq 0$. Then $C_n(a,b)$ is non-planar.*

Proof. The assumptions and the uniqueness of the associated HNF imply that $x_{11}a \equiv -x_{21}b \not\equiv 0 \pmod{n}$ and $2 \leq x_{11} \leq x_{21} \leq x_{22}/2$. We construct a subgraph G' in Table 4.4. The vertices

$$
\begin{aligned}
&0 \leftrightarrow a,\\
&0 \leftrightarrow b,\\
&0 \leftrightarrow -a \equiv (x_{11}-1)a+x_{21}b \leftrightarrow (x_{11}-2)a+x_{21}b \leftrightarrow \ldots \leftrightarrow 2a+x_{21}b \leftrightarrow a+x_{21}b,\\
&a+(x_{22}-1)b \leftrightarrow a,\\
&a+(x_{22}-1)b \leftrightarrow (x_{22}-1)b \leftrightarrow (x_{22}-2)b \leftrightarrow \ldots \leftrightarrow 2b \leftrightarrow b,\\
&a+(x_{22}-1)b \leftrightarrow a+(x_{22}-2)b \leftrightarrow a+(x_{22}-3)b \leftrightarrow \ldots \leftrightarrow a+(x_{21}+1)b \leftrightarrow a+x_{21}b,\\
&a+b \leftrightarrow a,\\
&a+b \leftrightarrow b,\\
&a+b \leftrightarrow a+2b \leftrightarrow a+3b \leftrightarrow \ldots \leftrightarrow a+(x_{21}-1)b \leftrightarrow a+x_{21}b.
\end{aligned}
$$

TABLE 4.4. Lemma 4.5

$2a+x_{21}b, \ldots, (x_{11}-1)a+x_{21}b; 2b, \ldots, (x_{22}-1)b; a+(x_{21}+1)b, \ldots, a+(x_{22}-2)b; a+2b, a+3b, \ldots a+(x_{21}-1)b$ have degree 2 in G'. After removing these vertices, we end up with an instance of $K_{3,3}$. □

Summing up Lemmata 4.2, 4.3, 4.4, and 4.5 and translating the conditions on the HNF into conditions on a and b taking into account that $\gcd(a,b,n)=1$, we get the results for $m=2$ which have been claimed in Theorem 4.1.

Since the proof of the cases $m \geq 3$ necessitates a characterization for the planarity of not necessarily connected graphs, we restate the result:

Proposition 4.6. *Let $C_n(a,b)$ be a properly given circulant which may be disconnected. It is planar if and only if one of the following two conditions holds:*

(13a) $a_i \equiv \pm 2a_j \pmod{n}$ *and* $v_2(a_j) < v_2(n)$, *where* $(i,j)=(1,2)$ *or* $(i,j)=(2,1)$.
(13b) $a_i = n/2$, $1 \leq v_2(n) \leq v_2(a_j)$, *where* $(i,j)=(1,2)$ *or* $(i,j)=(2,1)$.

4.2. **Circulant Graphs with $m \geq 3$ are Non-Planar.**

Lemma 4.7. *Let $C_n(a,b,c)$ be a properly given connected circulant. Then it is non-planar.*

Proof. Let us first assume that none of a, b, and c equals $n/2$. If $C_n(a,b,c)$ is planar, then $C_n(a,b)$ is planar, and this would imply by Proposition 4.6 that $a \equiv 2b \pmod{n}$ (interchanging a and b or replacing a by $n-a$ if necessary) and $v_2(b) < v_2(n)$. Similarly we get $b \equiv \pm 2c$ or $c \equiv \pm 2b$. The latter can be excluded since the circulant was properly given. Therefore (replacing c by $n-c$ if necessary) $b \equiv 2c \pmod{n}$ and $v_2(c) < v_2(n)$. This implies that $C_n(a,b,c) = C_n(4c,2c,c)$ with $v_2(2c) < v_2(n)$, i. e., $v_2(c) \le v_2(n) - 2$. It follows that $C_n(4c,c)$ is planar and we conclude from Proposition 4.6 that $n \mid uc$ for $u \in \{2,6,7,9\}$. This yields $v_2(n) \le 1 + v_2(c) \le 1 + (v_2(n) - 2)$, which is a contradiction.

We assume now that one of a, b, and c equals $n/2$, without loss of generality $c = n/2$. Since $C_n(a,n/2)$ is planar, either $1 \le v_2(n) \le v_2(a)$ by (13b) or $a \equiv \pm n/4 \pmod{n}$ by (13a). The same holds for b. As $C_n(a,b)$ is planar, either $v_2(b) < v_2(n)$ or $v_2(a) < v_2(n)$ by (13a), therefore we may assume without loss of generality that $C_n(a,b,c) = C_n(a,n/4,n/2)$ with $1 \le v_2(n) \le v_2(a)$. By (13a) we get $a \equiv \pm 2(n/4) \pmod{n}$ and have $a = n/2$, which is a contradiction. □

It is clear that Lemma 4.7 completes the proof of Theorem 4.1 since for $m \ge 4$, $C_n(a_1,a_2,a_3)$ is a non-planar subgraph of $C_n(a_1,a_2,\ldots,a_m)$.

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