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AUTOMATIC SOLUTION OF FAMILIES OF THUE EQUATIONS AND AN EXAMPLE OF DEGREE 8

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Abstract. We describe a procedure implemented in Mathematica® to solve parametrized families of Thue equations $F_n(X,Y) = \pm 1$, where F_n is a binary irreducible form in X and Y of degree $d \geq 3$ whose coefficients are polynomials in the parameter n. This procedure uses Baker's method and asymptotic expansions of the quantities involved with exact remainder terms. As an example, we solve a family of degree 8.

1. Introduction

Let $F \in \mathbb{Z}[X,Y]$ be a binary irreducible form of degree $d \geq 3$ and $m \neq 0$ an integer. The Diophantine equation

$$F(x,y) = m$$

is called a Thue equation, remembering that A. Thue [Thu09] proved that it only has finitely many solutions in integers x, y. Nowadays, the solution of a single Thue equation can be found algorithmically using Baker's method [Bak68] and reduction techniques to reduce the usually big upper bound coming from the linear form estimates, cf. Bilu and Hanrot [BH96].

In 1990, E. Thomas [Tho90] first considered a parametrized family of Thue equations with positive discriminant. Since that time, several such families $F_n(X,Y) = m$ have been solved, where $F_n \in \mathbb{Z}[n][X,Y]$ is a binary irreducible form in X and Y whose coefficients are polynomials in the parameter n, cf. for instance [Heu00] or Wakabayashi [Wak02b] or the online survey [Heu]. In all these families, there were some polynomial solutions $(x,y) \in \mathbb{Z}[n] \times \mathbb{Z}[n]$ such that $F_n(x,y) = m$ holds in $\mathbb{Z}[n]$ and therefore for every specialization of n to a concrete integer. Moreover, there were finitely many extra solutions (x,y) for finitely many values of the parameter n. We will call these solutions "sporadic solutions". In all papers, a bound n_0 has been found such that for $n \geq n_0$, there are no sporadic solutions. If n_0 was small enough, all sporadic solutions could also be found by solving each of the equations for the remaining values of n separately.

Many of these families have been solved by using Baker's method combined with direct arguments to exclude solutions of small and medium size. In these cases, the explicit calculations used for single equations have been replaced by asymptotic calculations involving the parameter n. Whereas these calculations can be done easily for equations of small degree, this becomes more or less impossible to do by hand for larger degrees: it is not sufficient to know the leading terms of the asymptotic expansions, but the knowledge of an explicit error bound is also necessary.

The aim of the present paper is to provide a procedure to do these calculations automatically. It has been implemented in Mathematica® and is available at `http://finanz.math.tu-graz.ac.at/~ziegler/Publications/AutomaticSolutionofThueEq`. We describe the general framework and the principles used in the implementation of the routines. We address some particular technical problems which arise and their solutions in our package. Of course, we cannot expect to solve every given family

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of Thue equations: it is known (cf. Lettl [Let]) that there are families of Thue equations which have infinitely many sporadic solutions. We do not make any attempt to implement the solution via Padé approximations, although they have been used for the solution of some families, too.

We demonstrate the use of our routines by solving a family of Thue equations of degree 8. To our knowledge, it is the first time that a family of degree > 6 is solved. Furthermore, we reconsider some families which have been solved previously "by hand" and apply our machinery. At the present state, we cannot reach the best known constants n_0 (as defined above), since our routines do not yet implement all tricks used in the previous papers. In particular, we always use the linear form in d logarithms directly, whereas the number of logarithms can sometimes be reduced by a careful study. Since the constants in linear form estimates depend on the number of logarithms dramatically, the results can be improved.

For the family of degree 8, the CPU time for the calculations was considerable (around 30 days on a Pentium 4 with 2 GHz running under Linux), which was also due to the fact that the coefficients of the asymptotic expansions were elements of $\mathbb{Q}(\sqrt{2})$. The example of degree 5, where the coefficients belong to $\mathbb{Q}$, can be solved in two to three hours. In the cubic case, it takes only a few minutes to get the result.

The remainder of the paper is organized as follows: In Section 2, we recall the general framework avoiding technical details as much as possible. Section 3 describes the implementation in more detail. The family of degree 8 is solved in Section 4. Section 5 is devoted to the known families of lower degree.

2. The procedure

We now give an outline of our procedure to solve parametrized families of Thue equations. We consider the Thue equation

$$F_n(X,Y) = \pm 1, \tag{1}$$

where $F_n \in \mathbb{Z}[n][X,Y]$ is an irreducible form of degree $d \geq 3$ and sufficiently large n. Let $f_n(X) := F_n(X,1)$ and denote the roots of f_n by $\alpha^{(1)},\dots,\alpha^{(d)}$. We assume that f_n is monic and all roots $\alpha^{(1)},\dots,\alpha^{(d)}$ are real. Let $K^{(k)} = \mathbb{Q}(\alpha^{(k)})$ be the number field generated by $\alpha^{(k)}$ and let $\mathfrak{o}_{K^{(k)}}$ be its ring of algebraic integers ($1 \leq k \leq d$). We call a solution (x,y) to equation (1) trivial if $|y| \leq 1$.

Let $\eta_1^{(1)},\dots,\eta_r^{(1)}$ with $r = d-1$ be a system of independent units in $\mathfrak{o}_{K^{(1)}}$, then let $\eta_i^{(k)}$ denote the k-th conjugate of $\eta_i^{(1)}$ ($1 \leq i \leq r$, $1 \leq k \leq d$). Obviously $\eta_1^{(k)},\dots,\eta_r^{(k)}$ is a system of independent units in $\mathfrak{o}_{K^{(k)}}$ ($1 \leq k \leq d$). We assume $\log|\eta_i^{(k)}| \ll \log n$, where $g \ll h$ means that there is some effectively computable constant c such that $|g| < c \cdot h$.

Let (x,y) be a solution to (1) and choose $1 \leq j \leq d$ such that

$$\left|x - \alpha^{(j)}y\right| = \min_i |x - \alpha^{(i)}y|.$$

We say that (x,y) is a solution of type j and we define $\beta^{(k)} := x - \alpha^{(k)}y$ ($1 \leq k \leq d$). So equation (1) can be rewritten as

$$F_n(x,y) = (x-\alpha^{(1)}y)\cdots(x-\alpha^{(d)}y) = \beta^{(1)}\cdots\beta^{(d)} = N_{\mathbb{Q}}^{K^{(k)}}\left(\beta^{(k)}\right) = \pm 1, \tag{2}$$

where $N_{\mathbb{Q}}^{K^{(k)}}$ denotes the norm.

Then we have

$$|y|\left|\alpha^{(i)} - \alpha^{(j)}\right| = \left|(x-\alpha^{(i)}y) - (x-\alpha^{(j)}y)\right| \leq \left|x-\alpha^{(i)}y\right| + \left|x-\alpha^{(j)}y\right| \leq \left|2\beta^{(i)}\right|$$

for $i \neq j$. This implies together with equation (2)

$$|\beta^{(j)}| \leq \frac{2^{d-1}}{|y|^{d-1}\prod_{i\neq j}|\alpha^{(i)}-\alpha^{(j)}|} = \frac{2^{d-1}}{|y|^{d-1}|f_n'(\alpha^{(j)})|}. \tag{3}$$

We further assume $|f_n'(\alpha^{(j)})\cdot(\alpha^{(i)}-\alpha^{(j)})| \gg n$ for $i \neq j$. This implies

$$\begin{aligned} \log\left|\beta^{(i)}\right| &= \log\left|x-\alpha^{(j)}y-(\alpha^{(i)}-\alpha^{(j)})y\right| \\ &= \log|y| + \log\left|\alpha^{(i)}-\alpha^{(j)}\right| + \log\left(1-\frac{\beta^{(j)}}{y(\alpha^{(i)}-\alpha^{(j)})}\right) \\ &= \log|y| + \log\left|\alpha^{(i)}-\alpha^{(j)}\right| - \frac{\beta^{(j)}}{y}\cdot\frac{1}{\alpha^{(i)}-\alpha^{(j)}} + O\left(\frac{1}{n^2}\right) \qquad (4) \end{aligned}$$

for $i \neq j$.

Since $\beta^{(k)}$ is a unit by (2), there are integers $b_1,\dots,b_r$ and I with

$$I \leq \left[\mathfrak{o}_{K^{(k)}}^{\times} : \langle -1, \eta_1^{(k)},\dots,\eta_r^{(k)}\rangle\right]$$

such that

$$\log\left|\beta^{(k)}\right| = \frac{b_1}{I}\log\left|\eta_1^{(k)}\right| + \cdots + \frac{b_r}{I}\log\left|\eta_r^{(k)}\right|, \quad k \neq j. \tag{5}$$

Solving this system of linear equations by Cramer's rule we obtain

$$R\frac{b_k}{I} = u_k\log|y| + v_k - \frac{\beta_j}{y}w_k + r_k \tag{6}$$

for $1 \leq k \leq r$, where

$$\begin{aligned} u_k &= \det\left(\log\left|\eta_1^{(i)}\right|,\dots,\log\left|\eta_{k-1}^{(i)}\right|, 1, \log\left|\eta_{k+1}^{(i)}\right|,\dots,\log\left|\eta_r^{(i)}\right|\right)_{i\neq j}, \\ v_k &= \det\left(\log\left|\eta_1^{(i)}\right|,\dots,\log\left|\eta_{k-1}^{(i)}\right|, \log\left|\alpha^{(i)}-\alpha^{(j)}\right|, \log\left|\eta_{k+1}^{(i)}\right|,\dots,\log\left|\eta_r^{(i)}\right|\right)_{i\neq j}, \\ w_k &= \det\left(\log\left|\eta_1^{(i)}\right|,\dots,\log\left|\eta_{k-1}^{(i)}\right|, \frac{1}{\alpha^{(i)}-\alpha^{(j)}}, \log\left|\eta_{k+1}^{(i)}\right|,\dots,\log\left|\eta_r^{(i)}\right|\right)_{i\neq j}, \\ r_k &= \det\left(\log\left|\eta_1^{(i)}\right|,\dots,\log\left|\eta_{k-1}^{(i)}\right|, O\left(\frac{1}{n^2}\right), \log\left|\eta_{k+1}^{(i)}\right|,\dots,\log\left|\eta_r^{(i)}\right|\right)_{i\neq j} = O\left(\frac{\log^{r-1} n}{n^2}\right), \\ R &= \det\left(\log\left|\eta_1^{(i)}\right|,\dots,\log\left|\eta_{k-1}^{(i)}\right|, \log\left|\eta_k^{(i)}\right|, \log\left|\eta_{k+1}^{(i)}\right|,\dots,\log\left|\eta_r^{(i)}\right|\right)_{i\neq j}. \end{aligned}$$

In the next section we will compute the value of $O(1/n^2)$ in r_k more explicitly. We take some constant integers $\lambda_0,\lambda_1,\dots,\lambda_r$ and consider

$$\overline{b} := \lambda_0 I + \sum_{k=1}^r \lambda_k b_k, \qquad \overline{u} := \sum_{k=1}^r \lambda_k u_k, \qquad \overline{v} := \lambda_0 R + \sum_{k=1}^r \lambda_k v_k,$$

$$\overline{w} := \sum_{k=1}^r \lambda_k w_k, \qquad \overline{r} := \sum_{k=1}^r \lambda_k r_k.$$

From (6) we deduce that

$$R\frac{\overline{b}}{I} = \overline{u}\log|y| + \overline{v} - \frac{\beta^{(j)}}{y}\cdot\overline{w} + \overline{r}. \tag{7}$$

We try to choose $\lambda_0, \dots, \lambda_r$ in such a way that

$$\frac{1}{n} \ll \overline{u} \ll \frac{\log^{r-1} n}{n}$$
$$\overline{v} \ll \frac{\log^{r-1} n}{n}$$
$$\operatorname{sign}(\overline{u}) = \operatorname{sign}\left(\overline{u}\log y_0 + \overline{v} - \left|\frac{\overline{w}}{2f_n'(\alpha^{(j)})}\right| + \overline{r}\right) = 1, \tag{8}$$

where y_0 is a lower bound for nontrivial $|y|$ ($|y| > 1$). From the definition one can always choose $y_0 = 2$. In Section 3.5, we will discuss how to obtain a better lower bound y_0 for $|y|$. For $|y| \geq y_0$ this implies

$$R\frac{\overline{b}}{I} > 0,$$

hence $|\overline{b}| \geq 1$. By a theorem of Friedman [Fri89] we further obtain

$$\frac{|R|}{I} \geq \frac{|R|}{[\mathfrak{o}_{K^{(k)}}^{\times} : \langle -1, \eta_1^{(k)}, \dots, \eta_r^{(k)} \rangle]} = \frac{|R|}{|R|/\operatorname{Reg}(K^{(k)})} = \operatorname{Reg}(K^{(k)}) > 0.2$$

and so $I \leq 5 \cdot |R|$. From this inequality we also obtain $|\frac{R\overline{b}}{I}| > 0.2$.

Using (7), we solve $|R\overline{b}/I| > 0.2$ for $\log|y|$ and we obtain

$$\log|y| \gg \frac{n}{\log^{r-1} n}, \tag{9}$$

if y is nontrivial ($|y| > 1$).

Let $H(n)$ be an upper bound for the coefficients of F_n. Since the coefficients of $F_n(X,Y)$ are polynomials in n, we have $\log H(n) \ll \log n$. Since we assume $\log\left|\eta_i^{(k)}\right| \ll \log n$ we obtain $\operatorname{Reg}(K^{(k)}) \ll \log n$. Using a theorem of Bugeaud and Győry [BG96] we obtain

$$\log|y| \ll \log^{2r} n \cdot \log\log n, \tag{10}$$

a contradiction to (9). So we have $n \ll 1$. Since all bounds are effectively computable one can give an explicit bound n_0, such that (1) has only solutions (x,y) of type j with $|y| \leq 1$ for $n \geq n_0$. We will compute n_0 in the next section.

The upper bound obtained from the theorem of Bugeaud and Győry can be improved using Baker's method directly. From (3) we get

$$\left|x - \alpha^{(j)} y\right| \leq \frac{c_1}{|y|^{d-1}},$$

with

$$c_1 = \left|\frac{2^{d-1}}{f_n'(\alpha^{(j)})}\right|.$$

The $c_1, \dots$ are all effectively computable constants depending on $n, \alpha^{(k)}$ and $\eta_i^{(k)}$ for $1 \leq k \leq d, 1 \leq i \leq r$. From this inequality we obtain

$$\operatorname{sign}(y)\alpha^{(j)} - \frac{c_1}{|y|^d} < \frac{x}{|y|} < \operatorname{sign}(y)\alpha^{(j)} + \frac{c_1}{|y|^d}, \tag{11}$$

hence

$$y \cdot \left(\alpha^{(j)} - \alpha^{(i)}\right) - \frac{c_1}{|y|^{d-1}} < \beta^{(i)} < y \cdot \left(\alpha^{(j)} - \alpha^{(i)}\right) + \frac{c_1}{|y|^{d-1}}. \tag{12}$$

Putting $B := \max |b_i|$ and solving (5) with Cramer's rule we get the estimate

$$\frac{B}{I} \leq \frac{r \cdot \max_{i\neq j}(\Delta_i^{(j)}) \cdot \max_{i\neq j}(\left|\log|\beta^{(i)}|\right|)}{R} \leq c_2 \max_{i\neq j} \left|\log|\beta^{(i)}|\right|, \tag{13}$$

where

$$\Delta_i^{(j)} = \left|(\log|\eta_k^{(l)}|)_{k\neq i, l\neq j}\right|$$

are the cofactors of R. The $\Delta_i^{(j)}$ can be estimated by Hadamard's inequality:

Lemma 1. *Let $A = (a_{ij})_{1\leq i,j,\leq n}$ a $(n \times n)$-matrix with real entries then*

$$(\det A)^2 \leq \prod_{j=1}^n \sum_{i=1}^n (a_{ij})^2.$$

Using (12) and (13) one obtains

$$\frac{B}{I} \leq c_3 \log|y|, \tag{14}$$

with

$$c_3 = c_2 \cdot \max_{i\neq j} \left(1 + \frac{\log\left(\left|\alpha^{(j)} - \alpha^{(i)}\right| + \frac{c_1}{y_0^d}\right)}{\log y_0}\right).$$

For $k \neq l \in \{1,2,\ldots,d\} \setminus \{j\}$ one obtains by Siegel's identity and (12)

$$\left|1 - \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \frac{\beta^{(l)}}{\beta^{(k)}}\right| = \left|\frac{\alpha^{(k)} - \alpha^{(l)}}{\alpha^{(j)} - \alpha^{(l)}} \frac{\beta^{(j)}}{\beta^{(k)}}\right| \leq \frac{c_4}{|y|^d}, \tag{15}$$

with

$$c_4 = \left|\frac{\alpha^{(k)} - \alpha^{(l)}}{\alpha^{(j)} - \alpha^{(l)}}\right| \frac{c_1}{\left|\alpha^{(j)} - \alpha^{(k)}\right| - \frac{c_1}{y_0^d}}.$$

Next we will use a theorem of Matveev [Mat00, Corollary 2.3].

Lemma 2. *Denote by $\alpha_1, \ldots, \alpha_n$ algebraic numbers, not 0 or 1, by $\log\alpha_1, \ldots, \log\alpha_n$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and by $b_1, \ldots, b_n$ rational integers. Define $B = \max\{|b_1|, \ldots, |b_n|\}$, and $A_i = \max\{Dh(\alpha_i), |\log\alpha_i|, 0.16\}$ $(1 \leq i \leq n)$, where $h(\alpha)$ denotes the absolute logarithmic Weil height of α. Assume that the number $\Lambda = b_1\log\alpha_1 + \cdots + b_n\log\alpha_n$ does not vanish; then*

$$|\Lambda| \geq \exp\{-C(n,\varkappa)D^2 A_1 \cdots A_n \log(eD)\log(eB)\},$$

where $\varkappa = 1$ if $\mathbb{K} \subset \mathbb{R}$ and $\varkappa = 2$ otherwise and

$$C(n,\varkappa) = \min\left\{\frac{1}{\varkappa}\left(\frac{1}{2}en\right)^{\varkappa} 30^{n+3} n^{3.5}, 2^{6n+20}\right\}.$$

Applying this theorem to

$$\Lambda = I \log\left|\frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}}\right| + u_1 \log\left|\frac{\eta_1^{(l)}}{\eta_1^{(k)}}\right| + \cdots + u_r \log\left|\frac{\eta_r^{(l)}}{\eta_r^{(k)}}\right| \tag{16}$$

and using the estimate $|\log x| < 2|x-1|$ for $|x-1| \le \frac{1}{3}$ together with (14) and (15) it results

$$\begin{aligned}\exp(-\log I - c_5 \log(eB)) &\le \left|\log\left|\frac{\alpha^{(j)}-\alpha^{(k)}}{\alpha^{(j)}-\alpha^{(l)}}\right| + \frac{u_1}{I}\log\left|\frac{\eta_1^{(l)}}{\eta_1^{(k)}}\right| + \cdots + \frac{u_r}{I}\log\left|\frac{\eta_r^{(l)}}{\eta_r^{(k)}}\right|\right| \\ &\le \frac{2c_4}{|y|^d} = \exp\left(c_6 - c_7\frac{B}{I}\right),\end{aligned}$$

where c_5 comes from the theorem of Matveev (Lemma 2), $c_6 = \log 2 + \log c_4$ and $c_7 = \frac{d}{c_3}$, for $B > I$. From the inequality

$$\log I + c_5 + c_5 \log B > c_7 \frac{B}{I} - c_6 \tag{17}$$

one obtains an upper bound c_8 for B and by (13) and (4) an upper bound c_9 for $\log y$ with

$$c_9 = c_8 \cdot \left(\sum_{i=1}^{r} \left|\log|\eta_i^{(k)}|\right|\right) - \log\left(\left|\alpha^{(j)} - \alpha^{(k)}\right| - \frac{c_1}{2^d}\right).$$

The computation of the quantities $c_1, \ldots$ will be described in Section 3.4.

3. Implementation

This section describes the basic ideas of implementing the procedure described above.

3.1. "Exact O-Notation". One of the main problems is that the roots $\alpha^{(1)}, \ldots, \alpha^{(d)}$ are not known explicitly. But it suffices to know an asymptotic approximation of the roots. This can be done by some symbolic steps of Newton's method. In the following we use the L-notation. Let c be a real number, assume $f(x), g(x)$ and $h(x)$ are real functions and $h(x) > 0$ for $x > c$. We will write

$$f(x) = g(x) + L_c(h(x))$$

for

$$g(x) - h(x) \le f(x) \le g(x) + h(x).$$

The use of the L-notation is like the use of the O-notation but with the advantage to have an explicit bound for the error term. The following lemma is obvious from the definition and some power series expansions of elementary functions.

Lemma 3. *Let $h(x)$ and $g(x)$ be real functions and let $f(x), f_1(x)$ and $f_2(x)$ be non-negative real functions for $x > c, x > c_1$ and $x > c_2$ respectively. Then*

(1)
$$(h(x) + L_{c_1}(f_1(x))) + (g(x) + L_{c_2}(f_2(x))) = h(x) + g(x) + L_{\max(c_1,c_2)}(f_1(x) + f_2(x)).$$

(2)
$$\begin{aligned}(h(x) &+ L_{c_1}(f_1(x))) \cdot (g(x) + L_{c_2}(f_2(x))) \\ &= h(x)g(x) + L_{\max(c_1,c_2)}(|g(x)|f_2(x) + |h(x)|f_1(x) + f_1(x)f_2(x)).\end{aligned}$$

(3) *Assume $0 \le f(x) < h(x)$ if $x > c$, then*
$$\log(h(x) + L_c(f(x))) = \log(h(x)) + L_c\left(\frac{f(x)}{h(x) - f(x)}\right).$$

(4) *Assume $0 \le f(x) < |h(x)|$ for $x > c$, then*
$$\frac{1}{h(x) + L_c(f(x))} = \frac{1}{h(x)} + L_c\left(\frac{f(x)}{h(x)\cdot(h(x) - f(x))}\right).$$

For computing expressions (we want to compute determinants) with entries in L-notation it is useful to keep the L-term as simple as possible. We define:

Definition 1. *The quantity z is said to be given in simple L-form, if there are some $c \in \mathbb{R}$, $a, b \in \mathbb{Z}$ and $R(n, \log n)$ such that $z = R(n, \log n) + L(c \cdot n^a \cdot \log^b n)$.*

However, Lemma 3 does not give simple L-forms, so we have to simplify the results of Lemma 3 to that form.

Subroutine *(Simplify L-form)* : Given $g \in \mathbb{R}(X, Y)$, find $a, b \in \mathbb{Z}$ and $c, m \in \mathbb{R}$ such that

$$g(n, \log n) = L_m(c \cdot n^a \cdot \log^b n)$$

and the L-term has still the same order of magnitude as g.

To find a, b, c is rather easy. Let

$$g(X, Y) = \frac{f(X, Y)}{h(X, Y)},$$

where $f(X, Y)$ and $h(X, Y)$ are polynomials. Let $f_1 = c_1 X^{a_1} Y^{b_1}$ and $h_1 = c_2 X^{a_2} Y^{b_2}$ be the monomials of highest degree (lexicographically) of $f(X, Y)$ and $h(X, Y)$ respectively, further let $f_2 = c_3 X^{a_3} Y^{b_3}$ be the monomial of highest degree $\neq f_1$ of f such that $\operatorname{sign} c_3 = \operatorname{sign} c_1$ (if no such monomial exists set $c_3 = 0$). Then set $a = a_1 - a_2$, $b = b_1 - b_2$, $c' = \frac{|c_1| + |c_3|}{|c_1|}$ and $c = c'|c_1/c_2|$. In practice one will get numerical problems to calculate m if $(c_1 + c_3)/c_1$ is too close to 1. So we will set $c' = \max(1.1, (|c_1| + |c_3|)/|c_1|)$.

To get m we have to find an upper bound for the real solutions of $cn^a \log^b n - g(n, \log n) = 0$ if such solutions exist, otherwise set $m = 0$. So we have reduced our problem to finding an upper bound for the largest root of $f(n, \log n) = 0$ for some given polynomial f. We will use two routines to get that upper bound.

(1) We will substitute $\log n = q$ and treat q as an independent variable. Let $p = 1 + \deg_q f$, let $f_i(q)$ be the coefficient of n^i and let d_i be the leading coefficient of f_i. We will construct a new function $\overline{f}(n)$ such that $\overline{f}(n) \leq f(n, \log n)$ for $n \geq m_{-1}$ as follows:
   (a) If $d_i > 0$ let m_i' be the largest real solution of $f_i(q) = d_i$ (if it doesn't exist set $m_i = 0$), set $\overline{f}_i := d_i$ and $m_i = \exp(m_i')$.
   (b) If $d_i < 0$ let m_i' be the largest real solution of $\log n = n/p$ if such exist (which is the case for $p \geq 3$) and set $m_i' = 0$ otherwise. Set all coefficients in $f_i(q)$ which are > 0 to 0 and substitute $q = n^{1/p}$. Set $m_i = m_i'^p$.

   Let $\overline{f}(n)$ be the function obtained by these substitutions and let $m_{-1} = \max_i(m_i)$, then for all $n > m_{-1}$ we have

$$\bar{f}(n) \leq f(n, \log n) \quad \text{for} \quad n \geq m_{-1}.$$

   Let m_{-2} be an upper bound for the largest real root of $\bar{f}$ then $m = \max(m_{-1}, m_{-2})$ is an upper bound for n such that $f(n, \log n) = 0$. We can compute m_{-2}, since the substitution $n \to n^p$ transforms $\bar{f}$ into a polynomial. Since there are algorithms (as implemented in Mathematica or Pari) to find all roots, in particular the largest real root of a polynomial, we are done.

(2) Similar to the first routine we will construct a function $\tilde{f}(n)$ such that $\tilde{f}(n) \leq f(n, \log n)$ for $n \geq m_0$ with $q = \log n$. To obtain $\tilde{f}(n)$ we set all coefficients of f that are positive to 0 except the leading term (in lexicographical order). Then we substitute $q = n^{1/p}$ where $p = \deg_q f$. Let m_0' be the largest real root of $\log n = n/p$ (if no real root exists let $m_0' = 0$) and let $m_0 = m_0'^p$. We obtain

$$\tilde{f}(n) \leq f(n, \log n) \quad \text{for} \quad n \geq m_0.$$

Let m_1 be the largest real root of $\tilde{f}(n) = 0$ then $m = \max(m_0, m_1)$ is an upper bound for the root n of $f(n, \log n) = 0$.

Now we take the minimum of the bounds obtained by these two routines.

Next we want to obtain a simple L-form of the logarithm of a simple L-form. This can be done by using Lemma 3 and the following result.

Lemma 4.

$$\log(R(n) + L(c/n^k)) = a \cdot q + \log b + Q(n) + L\left(\frac{d}{n^l}\right),$$

where bn^a is the main part of the Laurent expansion at ∞ of $R(n), Q(n) \in \mathbb{R}[n, 1/n]$ with $Q(n) = o(1)$, $l = a + k$ and d is some effectively computable constant depending on R.

Proof: Using the power series for log and Lemma 3 one obtains

$$\log\left(R(n) + L(c/n^k)\right) = \log(R(n)) + L\left(\frac{c}{n^k R(n) - c}\right) = \log(bn^a + T(n)) + L(c_1/n^{a+k})$$

with $T(n) = R(n) - bn^a = o(R(n))$. The power series expansion of $\log R(n)$ at cn^a gives the lemma.

■

3.2. **Calculation of the necessary quantities in simple L-form.** We will use the index of the L-notation only for concrete computations and will omit it for brevity in most cases. We assume that the roots $\alpha^{(j)}$ are given in simple L-form. We further assume a system of independent units $\eta_1^{(i)}, \dots, \eta_r^{(i)}$ is given by rational functions $R_i(x)$ such that $R_i(\alpha^{(1)}) = \eta_i^{(1)}$. Using Lemma 3 and the described procedure to get simple L-terms, we can easily compute all units $\eta_i^{(1)}$ and their conjugates in simple L-form.

Using Lemma 4 one gets the matrix

$$\begin{pmatrix} \log\left|\eta_1^{(1)}\right| & \dots & \log\left|\eta_r^{(1)}\right| \\ \vdots & \ddots & \vdots \\ \log\left|\eta_1^{(d)}\right| & \dots & \log\left|\eta_r^{(d)}\right| \end{pmatrix},$$

where the entries are given in simple L-form. Given the type j of the solution a similar computation gives the matrices considered in a quantitative form and hence the determinants R, u_k, v_k and w_k for $1 \le k \le d$.

Next we want to compute the determinants r_k $(1 \le k \le r)$. Since $|y| \ge 2$ and

$$\frac{\beta^{(j)}}{y} = L\left(\frac{2^{d-1}}{\left|y^d f_n'(\alpha^{(j)})\right|}\right) = L\left(\frac{1}{2|f_n'(\alpha^{(j)})|}\right),$$

by (3) and

$$\log(1+x) = x - \frac{1}{2}x^2 + \cdots = x + L\left(\frac{1}{2}x^2(1 + x + \cdots)\right) = x + L\left(\frac{1}{2}\frac{x^2}{1-|x|}\right)$$

for $|x| < 1$ we obtain

$$\log\left|\beta^{(i)}\right| = \log|y| + \log\left|\alpha^{(i)} - \alpha^{(j)}\right| - \frac{\beta^{(j)}}{y}\cdot\frac{1}{\alpha^{(i)}-\alpha^{(j)}} + L\left(\frac{1}{4\left(2{f_n'}^2(\alpha^{(j)})\cdot(\alpha^{(i)}-\alpha^{(j)})^2 - f_n'(\alpha^{(j)})\cdot(\alpha^{(i)}-\alpha^{(j)})\right)}\right) \tag{18}$$

for $|y| > |\frac{\beta^{(j)}}{\alpha^i - \alpha^j}|$. With (18) we can compute the determinants r_k $(1 \le K \le r)$ in simple L-form.

3.3. **Computation of the lower bound.** The computation of the λ's can be done by solving the equations obtained by comparing coefficients. Once the λ's are computed it is easy to obtain a lower bound for $\log|y|$ by solving the inequality

$$0.2 < \frac{R}{I} \le \overline{u}\log|y| + \overline{v} - \overline{w}\cdot\frac{\beta_j}{y} + \overline{r}$$

obtained from (7). Since $\frac{\beta_j}{y}$ appears as coefficient of $\overline{w}$, we use the estimate $\left|\overline{w}\cdot\frac{\beta_j}{y}\right| < \left|\frac{\overline{w}}{2f_n'(\alpha^{(j)})}\right|$ to compute a lower bound for $\log|y|$.

3.4. **Calculation of the upper bound.** Computing c_8 amounts essentially to solve an equation of the form $h(x) := cx - a - \log x = 0$. This can be done by using one step of Newton's method starting at $x_0 > 1/c$, since $h''(x) > 0$ and $h'(x) > 0$ for $x > 1/c$. Hence applying one step of Newton's method will give an upper bound for the root x of $h(x) = 0$.

Lemma 5. *Let $a, c \in \mathbb{R}^+$, $0 < \varepsilon < 1$ and $x > 0$. Then for all*

$$x > \left(a - \frac{1}{\varepsilon} - \log c\varepsilon\right)\frac{1}{(1-\varepsilon)c} + \frac{1}{c\varepsilon}$$

we have $h(x) = cx - a - \log x > 0$.

Proof: Apply one step of Newton's method starting at the point $x_0 = \frac{1}{\varepsilon c}$. ∎

In the implementation we used the value $\varepsilon = \frac{1}{10}$.

3.5. **Finding "trivial" lower bounds for $|y|$.** Assume (x, y) is a nontrivial ($|y| > 1$) solution of type j of the Thue equation $F_n(X, Y) = \pm 1$ and

$$\alpha^{(j)} = P(n) + \frac{Q(n)}{n^k} + L\left(\frac{c}{n^{k+1}}\right)$$

with $P \in \mathbb{Z}[X]$, $Q \in \mathbb{R}[X]$, $\deg Q < k$, $c \in \mathbb{R}$ and $0 < k \in \mathbb{Z}$. From (11) we obtain

$$y\overbrace{\left(\frac{Q(n)}{n^k} - \frac{c}{n^{k+1}} - \frac{c_1}{2^d}\right)}^{y_1} < x - P(n)y < y\overbrace{\left(\frac{Q(n)}{n^k} + \frac{c}{n^{k+1}} + \frac{c_1}{2^d}\right)}^{y_2}.$$

Since $x - P(n)y$ is an integer we have $x - P(n)y = 0$, if

$$|y| < y_0 := \min\left(\left|\frac{1}{y_1}\right|, \left|\frac{1}{y_2}\right|\right). \tag{19}$$

Assume $x = P(n)y$. Substitute $P(n)y$ for x in the Thue equation to obtain

$$y^d \cdot T(n) = \pm 1,$$

where $T(n) \in \mathbb{Z}[X]$. Hence the only possibility for y to satisfy this equation is $|y| = 1$ and since (x, y) is nontrivial we obtain $y \geq y_0$.

Wakabayashi [Wak02a] showed how to obtain further "trivial" bounds, using continued fraction expansions of $\alpha^{(j)}$ and a generalization of Legendre's Theorem. For details see [Wak02a], Section 6.

4. An equation of degree 8

As an example for the use of the procedure described above, we consider the parametrized familiy of Thue equations of degree 8

$$F_n(X,Y) := X^8-8nX^7Y-28X^6Y^2+56nX^5Y^3+70X^4Y^4-56nX^3Y^5-28nX^2Y^6+8nXY^7+Y^8 = \pm 1. \tag{20}$$

We first want to construct these polynomials in order to understand the structure. Let $\varepsilon := 1+\sqrt{2}$ and let

$$A = \begin{pmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{pmatrix}.$$

Then A is of order 8 in the group $\mathbf{PGL}_2(\mathbb{Q}(\sqrt{2}))$, since

$$A^2 \sim \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A^3 \sim \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad A^4 \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^5 \sim \begin{pmatrix} -1 & -\varepsilon \\ \varepsilon & -1 \end{pmatrix},$$
$$A^6 \sim \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A^7 \sim \begin{pmatrix} -\varepsilon & -1 \\ 1 & -\varepsilon \end{pmatrix}, \quad A^8 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the usual action of $\mathbf{PGL}_2(\mathbb{Q}(\sqrt{2})$ on $\mathbb{C}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Let $\vartheta_i = A^{i-1}\vartheta, i = 1, \dots, 8$. Writing them out, we have

$$\vartheta_1 = \vartheta, \quad \vartheta_2 = \frac{\varepsilon\vartheta - 1}{\vartheta + \varepsilon}, \quad \vartheta_3 = \frac{\vartheta - 1}{\vartheta + 1},$$
$$\vartheta_4 = \frac{\vartheta - \varepsilon}{\varepsilon\vartheta + 1}, \quad \vartheta_5 = \frac{-1}{\vartheta}, \quad \vartheta_6 = \frac{-\vartheta - \varepsilon}{\varepsilon\vartheta - 1}$$
$$\vartheta_7 = \frac{-\vartheta - 1}{\vartheta - 1}, \quad \vartheta_8 = \frac{-\varepsilon\vartheta - 1}{\vartheta - \varepsilon}$$

Since $\vartheta_i = \frac{-1}{\vartheta_{i+4}}, (i = 1, \dots, 4)$, we have $\prod_{i=1}^8 \vartheta_i = 1$. Therefore ϑ is a root of the octic polynomial

$$P(X) = X^8 - a_1X^7 + a_2X^6 - a_3X^5 + a_4X^4 - a_5X^3 + a_6X^2 - a_7X + 1,$$

where $a_1 = \sum_{i=1}^8 \vartheta_i$, $a_2 = \sum_{i<j} \vartheta_i\vartheta_j$ etc. Shen [She91] showed that

$$\begin{aligned} P(X) &= X^8 - a_1X^7 - 28X^6 + 7a_1X^5 + 70X^4 - 7a_1X^3 - 28X^2 + a_1X + 1 \\ &= X^8 - 28X^6 + 70X^4 - 28X^2 + 1 - a_1X(X^2-1)(X^2-\varepsilon^2)(X^2-\varepsilon^{-2}). \end{aligned} \tag{21}$$

Since $P(\infty) > 0$ and $P(\varepsilon) < 0$ there is a real root of $P(X) = 0$. The construction of the polynomial shows that $P(X) = 0$ has eight distinct real roots satisfiying

$$\vartheta_1 \in (\varepsilon, \infty), \quad \vartheta_2 \in (1, \varepsilon), \quad \vartheta_3 \in (\varepsilon^{-1}, 1),$$
$$\vartheta_4 \in (0, \varepsilon^{-1}), \quad \vartheta_5 \in (-\varepsilon^{-1}, 0), \quad \vartheta_6 \in (-1, -\varepsilon^{-1}),$$
$$\vartheta_7 \in (-\varepsilon, -1), \quad \vartheta_8 \in (-\infty, -\varepsilon).$$

They are all units in the ring of algebraic integers of the field $\mathbb{Q}(\vartheta, \sqrt{2})$, if a_1 is an algebraic integer of the field $\mathbb{Q}(\sqrt{2})$. Shen could prove the following proposition (Proposition 1 in [She91]).

Proposition 1. *The octic polynomial $P(X)$ in equation (21) is irreducible over the field $\mathbb{Q}$ for $a_1 \in \mathbb{Z} \setminus \{0, \pm 6, \pm 15\}$.*

Let $y := \frac{1}{2}(\vartheta_1 + \vartheta_5)$, $z := \frac{1}{4}(\vartheta_1 + \vartheta_3 + \vartheta_5 + \vartheta_7)$ and let $8n = a_1$, (with this substitution we get $f_n(X)$) then Shen proved:

Proposition 2. (1) *The minimal polynomial of y over $\mathbb{Q}$ is*

$$X^4 - 4nX^3 - 6X^2 + 4nX + 1,$$

and hence $\mathbb{Q}(y)$ is a "simplest quartic field".

(2) *The minimal polynomial of z over $\mathbb{Q}$ is*

$$X^2 - 2nX - 1,$$

and hence $\mathbb{Q}(z) = \mathbb{Q}(\sqrt{n^2+1})$.

(3) *We have $\mathbb{Q}(z) = \mathbb{Q}(\sqrt{2})$ if $n \in S := \{a \in \mathbb{Z} \,:\, a + b\sqrt{2} = \varepsilon^{2n+1}, n \in \mathbb{N}\}$.*

(4) *For $n \in S$ the field $K_n := \mathbb{Q}(\rho)$ is a totally real cyclic octic field, whose Galois group $G(K_n|\mathbb{Q}) = <\sigma> \simeq \mathbb{Z}/8\mathbb{Z}$.*

(5) *The units $\rho_1, \rho_2, \rho_3, \rho_4, y_1, y_2$ and ε in the ring of algebraic integers $\mathfrak{o}_{K_n}$ are independent.*

(6) *The regulator R of K_n has the lower bound*

$$2^{-6} \log \varepsilon \log^6 n.$$

Since the algebraic data required for solving the family is known for $n \in S$ only, we will restrict our attention to this case. Let ρ be the largest root of $f_n(X) = 0$ and let $\rho_i = \sigma^{i-1}\rho$ for $i = 1, \dots, 8$, where $\sigma \in G(\mathbb{Q}(\rho)|\mathbb{Q})$ is determined by

$$\rho \mapsto \frac{\varepsilon\rho - 1}{\rho + \varepsilon}.$$

Since $n \in S$ we have $\sqrt{2} \in \mathbb{Q}(\rho)$ and hence $\varepsilon \in \mathbb{Q}(\rho)$ and so σ is indeed an automorphism. Note that $\rho_i = \vartheta_i$ for $i = 1, 2, 5, 6$ but

$$\rho_3 = \vartheta_7, \quad \rho_4 = \vartheta_8, \quad \rho_7 = \vartheta_3, \quad \rho_8 = \vartheta_4.$$

We have a different ordering since $<\sigma|_{\mathbb{Q}(z)}> = G(\mathbb{Q}(\sqrt{2})|\mathbb{Q})$ and hence $\sigma(\varepsilon) = -\varepsilon^{-1}$.

As above let $y := \frac{1}{2}(\rho_1 + \rho_5) = \frac{1}{2}(\rho - 1/\rho)$ and $z := \frac{1}{4}(\rho_1 + \rho_3 + \rho_5 + \rho_7) = \frac{1}{2}(y - 1/y)$. Hence we obtain two equations

$$\rho^2 - 2y\rho - 1 = 0 \quad \text{and} \quad y^2 - 2yz - 1 = 0.$$

It is easy to compute $\rho_i, (i = 1, \dots, 8)$ by solving these quadratic equations recursively. As $n \to \infty$ one obtains:

$$\rho_1 \sim 8n, \quad \rho_2 \to \varepsilon, \quad \rho_3 \to -1, \quad \rho_4 \to -\varepsilon,$$

$$\rho_5 \to 0, \quad \rho_6 \to -\varepsilon^{-1}, \quad \rho_3 \to 1, \quad \rho_4 \to \varepsilon^{-1}.$$

We apply Newton's method three times starting at $x_i = \lim_{n\to\infty} \rho_i(n)$ for $2 \le i \le 8$ and $x_1 = 8n$. We obtain:

$$\begin{aligned}
\rho_1 &= 8n + \frac{21}{8n} + L\left(\frac{1.05}{n^3}\right);\\
\rho_2 &= \varepsilon - \frac{\varepsilon}{2\sqrt{2}n} + \frac{\varepsilon^2}{16\sqrt{2}n^2} - L\left(\frac{0.19734}{n^3}\right);\\
\rho_3 &= -1 - \frac{1}{4n} - \frac{1}{32n^2} + L\left(\frac{0.083125}{n^3}\right);\\
\rho_4 &= -\varepsilon - \frac{\varepsilon}{2\sqrt{2}n} - \frac{\varepsilon^2}{16\sqrt{2}n^2} - L\left(\frac{0.20734}{n^3}\right);\\
\rho_5 &= -\frac{1}{8n} + L\left(\frac{0.036016}{n^3}\right);\\
\rho_6 &= -\varepsilon^{-1} - \frac{\varepsilon^{-1}}{2\sqrt{2}n} - \frac{\varepsilon^{-2}}{16\sqrt{2}n^2} - L\left(\frac{0.05267}{n^3}\right);\\
\rho_7 &= 1 - \frac{1}{4n} + \frac{1}{32n^2} + L\left(\frac{0.073125}{n^3}\right);\\
\rho_8 &= \varepsilon^{-1} - \frac{\varepsilon^{-1}}{2\sqrt{2}n} + \frac{\varepsilon^{-2}}{16\sqrt{2}n^2} - L\left(\frac{0.04267}{n^3}\right).
\end{aligned}$$

So all input is collected to use the procedure to solve the family $F_n(X,Y) = \pm 1$. We remark that the coefficients of the asymptotic expansions of the ρ_i contain ε and therefore $\sqrt{2}$. This makes the computation lengthier.

Before applying the procedure we collect some other useful facts. First we prove a lemma about the type of a solution.

Lemma 6. *If (x,y) is a solution of type $j = 1,2,3,4$, then $(y,-x)$ is a solution of type $j+4$ for $n > 1002$.*

Proof: We have $\rho_i = -1/\rho_{i+4}$. Using the L-form representation of the ρ_i we obtain

$$\min|\rho^{(i)} - \rho^{(j)}| > 0.3 \text{ for } i \neq j \text{ and } (n > 10).$$

We further obtain by computing c_1 from Section 2

$$|x/y - \rho^{(j)}| < \frac{1}{16n} < 6.25 \cdot 10^{-5} \text{ for } (n > 1002).$$

These two inequalities prove the lemma.

■

Since $y^{(i)} = y^{(i+4)}$ for $i = 1,2,3,4$ we can choose l and k from the linear form (16) such that $l = k+4$ and we obtain a linear form in only five logarithms. So we get upper bounds

type 1: $\log|y| < 4.339 \cdot 10^{27} \cdot \log^8 n$,
type 2: $\log|y| < 3.136 \cdot 10^{28} \cdot \log^8 n$,
type 3: $\log|y| < 4.339 \cdot 10^{27} \cdot \log^8 n$,
type 4: $\log|y| < 4.412 \cdot 10^{27} \cdot \log^8 n$,

Calculating the determinants from Section 2 we get lower bounds

type 1: $\log|y| > 4.8 \cdot 10^{-4} \cdot n \cdot \log^2 n$,
type 2: $\log|y| > 4.8 \cdot 10^{-4} \cdot n \cdot \log^2 n$,
type 3: $\log|y| > 2.4 \cdot 10^{-4} \cdot n \cdot \log^2 n$,
type 4: $\log|y| > 4.8 \cdot 10^{-4} \cdot n \cdot \log^2 n$,

Comparing these bounds we obtain a bound for n_0.

Theorem 1. *The Thue equation (20) has only trivial solutions for $n \geq n_0$ and $n \in S$ with $n_0 = 6.71 \cdot 10^{32}$.*

Looking at the structure of S one obtains

$$S = \left\{ a(n) := \frac{1}{2}\left((1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1}\right) \,:\, n \in \mathbb{N} \setminus \{0\} \right\}.$$

A quick computation shows that there are only 45 elements in S that are smaller than $3.4 \cdot 10^{34}$. A straight forward calculation shows:

Lemma 7. *Let $x, y, c \in \mathbb{Z}$.*

(1) *Suppose $F_n(x,y) = c$, then $F_n(x+y, -x+y) = F_n(x-y, x+y) = 16c$.*
(2) *Suppose $F_n(x,y) = 16c$, then $F_n(\frac{x+y}{2}, \frac{-x+y}{2}) = F_n(\frac{x-y}{2}, \frac{x+y}{2}) = c$.*

One observes that $F_n(x,y) = 16c$ implies that $x \equiv y \pmod 2$ and so $\frac{x+y}{2}$ and $\frac{x-y}{2}$ are integers. All together gives the following corollary.

Corollary 1. *Let $a(n) := \frac{1}{2}\left((1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1}\right)$ and let*

$$\begin{aligned} F_n(X,Y) := {} & X^8 - 8\,a(n)X^7Y - 28X^6Y^2 + 56\,a(n)X^5Y^3 + 70X^4Y^4 \\ & - 56\,a(n)X^3Y^5 - 28X^2Y^6 + 8\,a(n)XY^7 + Y^8. \end{aligned}$$

Let $n \geq 45$ then the Thue equation $F_n(X,Y) = c$ with $c \in \{\pm 1, \pm 16\}$ has only the integer solutions

$$\{(\pm 1, 0), (0, \pm 1)\} \text{ if } c = 1,$$

$$\{\pm(1,1), \pm(1,-1)\} \text{ if } c = 16$$

and there are no integer solutions for $c \in \{-1, -16\}$.

5. Further examples

In this section we will reconsider some other examples that have been solved before. In the particular cases, our results are worse than those obtained by other authors, since they used algebraic relations to reduce the linear form in logarithms (16) to a linear form in fewer logarithms. They also exploited the Galois group of the polynomial $f(X) = F(X,1)$ to get better estimates.

5.1. **The equation of Thomas and Mignotte.** We will now consider the Thue equation

$$X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 = \pm 1.$$

It has been solved for $n > 1.365 \cdot 10^7$ by Thomas [Tho90] and for all n by Mignotte [Mig93]. Let α be the largest root of

$$f_n(x) := F_n(X,1) = x^3 - (n-1)x^2 - (n+2)x - 1 = 0,$$

then $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a cyclic Galois extension and $\alpha, -1/(\alpha+1)$ are fundamental units of $\mathbb{Q}(\alpha)$. This was proved by Thomas [Tho79]. If (x,y) is a solution of type j, then $(y, -(x+y))$ is a solution of type $(j+1 \bmod 3)+1$. Hence it suffices to consider only one type. We treated the type 1. By using Newton's method we see that the roots of $f_n(x)=0$ are

$$\begin{aligned}
\alpha^{(1)} &= n+\frac{2}{n}-\frac{1}{n^2}+L\left(\frac{3}{n^3}\right),\\
\alpha^{(2)} &= -1-\frac{1}{n}+L\left(\frac{3}{n^3}\right),\\
\alpha^{(3)} &= -\frac{1}{n}+\frac{1}{n^2}+L\left(\frac{3}{n^3}\right).
\end{aligned}$$

Applying the procedure from Section 2 we obtain that there are only trivial solutions for $n>n_0$, where $n_0=4.13\cdot 10^{29}$. If we take into account the fact that $\alpha, -1/(\alpha+1)$ is a system of fundamental units, hence $I=1$, and that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is cyclic we get the better result $n_0=2.18\cdot 10^{20}$. The fact that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is cyclic leads to a better result since we know the structure of the Galois group, hence we can compute the quantity c_5 more effectively.

5.2. **An equation of degree 4.** The next example is the Thue equation

$$F_n(X,Y)=X^4-nX^3Y-X^2Y^2+nXY^3+Y^4=\pm 1.$$

This equation was first treated by Pethő [Pet91]. He proved that for $n>9.9\cdot 10^{27}$ there are only trivial solutions. The equation was solved in 1996 for $n\geq 3$ by Mignotte, Pethő and Roth. A system of fundamental units is given by $\alpha-1,\alpha,\alpha+1$, where α is the largest root of $f_n(X):=F_n(X,1)=0$. By Newton's method we obtain

$$\begin{aligned}
\alpha^{(1)} &= n-\frac{1}{n^3}+L\left(\frac{1}{n^4}\right),\\
\alpha^{(2)} &= -1+\frac{1}{2n}-\frac{1}{8n^2}+\frac{1}{2n^3}+L\left(\frac{1}{n^4}\right),\\
\alpha^{(3)} &= 1+\frac{1}{2n}+\frac{1}{8n^2}+\frac{1}{2n^3}+L\left(\frac{1}{n^4}\right),\\
\alpha^{(4)} &= -\frac{1}{n}+L\left(\frac{1}{n^4}\right).
\end{aligned}$$

If we use all information we have and use the procedure from Section 2 we obtain that there are only trivial solutions (x,y) for $n>n_0$, where $n_0=2.6\cdot 10^{34}$ if (x,y) is of type $j=1$. The other three cases $j=2,3,4$ do not satisfy the Assumption (8). Since $\alpha^{(2)}=R_2(n)+L(1/n^4)$, $\alpha^{(3)}=R_3(n)+L(1/n^4)$ and $\alpha^{(4)}=R_4(n)+L(1/n^4)$ with $R_2,R_3,R_4\in\mathbb{Z}(X)$ we can compute a "trivial" lower bound for $|y|$ and obtain (j is the type of solution):

(1) $\log|y|\geq \log n-1.4$ if $j=2$,
(2) $\log|y|\geq \log n-1.4$ if $j=3$,
(3) $\log|y|\geq \log n-2.1$ if $j=4$.

Using these "trivial" bounds we obtain

(1) $n_0=1.82\cdot 10^{35}$ if $j=2$,
(2) $n_0=8.49\cdot 10^{34}$ if $j=3$,
(3) $n_0=6.4\cdot 10^{34}$ if $j=4$.

Hence the Thue equation has only trivial solutions for $n\geq n_0$ with $n_0=1.82\cdot 10^{35}$.

5.3. **An equation of degree 5.** We now consider a Thue equation of degree 5:

$$X(X^2-Y^2)(X^2-n^2Y^2)+Y^5=\pm 1.$$

This equation was first solved by Heuberger [Heu98], who proved that there exist only trivial solutions for $n \geq n_0$ with $n_0 = 3.6 \cdot 10^{19}$. Using Newton's method we obtain:

$$\begin{aligned}
\alpha^{(1)} &= n + \frac{1}{2n^4} + \frac{1}{2n^6} + L\left(\frac{2}{n^7}\right),\\
\alpha^{(2)} &= -n + \frac{1}{2n^4} + \frac{1}{2n^6} + L\left(\frac{2}{n^7}\right),\\
\alpha^{(3)} &= 1 - \frac{1}{2n^2} - \frac{7}{8n^4} - \frac{5}{4n^6} + L\left(\frac{2}{n^7}\right),\\
\alpha^{(4)} &= -1 - \frac{1}{2n^2} - \frac{1}{8n^4} + \frac{3}{4n^6} + L\left(\frac{2}{n^7}\right),\\
\alpha^{(5)} &= \frac{1}{n^2} + \frac{1}{n^6} + L\left(\frac{2}{n^7}\right).
\end{aligned}$$

Heuberger could also prove that $\alpha^{(1)}, \alpha^{(1)}+1, \alpha^{(1)}-1, \alpha^{(1)}-n$ is a system of fundamental units. Using the procedure described in Section 2 we get that there are no non trivial solutions for $n \geq n_0$ with

(1) $n_0 = 5.09 \cdot 10^{43}$ if the solution is of type 1,
(2) $n_0 = 1.04 \cdot 10^{44}$ if the solution is of type 2,
(3) $n_0 = 4.6 \cdot 10^{44}$ if the solution is of type 3,
(4) $n_0 = 5.1 \cdot 10^{44}$ if the solution is of type 4.

For the type 5 the Assumption (8) is not true. But estimating a "trivial" lower bound for $|y|$ we get $\log|y| > 2\log n - 2.9$. Using this bound we can use the procedure described in Section 2 also in this case and obtain for solutions of type 5 that $n_0 = 5.71 \cdot 10^{44}$. Hence there are no trivial solutions of this equation for $n \geq n_0$ with $n_0 = 5.71 \cdot 10^{44}$.

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